Some matters of great balance

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Doctoral Thesis No. 144
Sundsvall, Sweden
2013

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SE-851 70 Sundsvall
SWEDEN

ISBN 978-91-87103-67-4
ISSN 1652-893X

Akademisk avhandling som med tillstånd av Mittuniversitetet framlägges till offentlig granskning för avläggande av filosofie doktorsexamen fredagen den 22 mars 2013 i O111, Mittuniversitetet, Holmgatan 10, Sundsvall.

Tryck: Tryckeriet Mittuniversitetet

To the Memory of My Parents
To My Children
To Liu

Abstract

This thesis is based on four papers dealing with two different areas of mathematics. Paper I–III are in combinatorics, while Paper IV is in mathematical physics.

In combinatorics, we work with design theory, one of whose applications are designing statistical experiments. Specifically, we are interested in *symmetric incomplete block designs* (SBIBDs) and *triple arrays* and also the relationship between these two types of designs.

In Paper I, we investigate when a triple array can be *balanced for intersection* which in the canonical case is equivalent to the *inner design* of the corresponding *symmetric balanced incomplete block design* (SBIBD) being balanced. For this we derive new existence criteria, and in particular we prove that the residual design of the related SBIBD must be quasi-symmetric, and give necessary and sufficient conditions on the intersection numbers. We also address the question of when the inner design is balanced with respect to every block of the SBIBD. We show that such SBIBDs must possess the *quasi*-3 property, and we answer the existence question for all know classes of these designs.

As triple arrays balanced for intersections seem to be very rare, it is natural to ask if there are any other families of row-column designs with this property. In Paper II we give necessary and sufficient conditions for *balanced grids* to be balanced for intersection and prove that all designs in an infinite family of *binary pseudo-Youden designs* are balanced for intersection.

Existence of triple arrays is an open question. There is one construction of an infinite, but special family called *Paley triple arrays*, and one general method for which one of the steps is unproved. In Paper III we investigate a third construction method starting from *Youden squares*. This method was suggested in the literature a long time ago, but was proven not to work by a counterexample. We show inter alia that Youden squares from projective planes can never give a triple array by this method, but that for every triple array corresponding to a biplane, there is a suitable Youden square for which the method works. Also, we construct the family of Paley triple arrays by this method.

In mathematical physics we work with solitons, which in nature can be seen as self-reinforcing waves acting like particles, and in mathematics as solutions of certain non-linear differential equations. In Paper IV we study the non-commutative version of the two-dimensional Toda lattice for which we construct a family of solutions, and derive explicit solution formulas.

keywords: Balanced incomplete block design. Triple array. Balanced grid. Pseudo-Youden design. Youden square. Inner balance. Balanced for intersection. Soliton. Two-dimensional Toda lattice.

Sammanfattning

Denna avhandling baseras på fyra artiklar som behandlar två olika områden av matematiken. Artikel I-III ligger inom kombinatoriken medan artikel IV behandlar matematisk fysik.

Inom kombinatoriken arbetar vi med designteori som bland annat har tillämpningar då man ska utforma statistiska experiment.

I artikel I undersöker vi när en *triple array* kan vara *snittbalanserad* vilket i det kanoniska fallet är ekvivalent med den *inre designen* till den korresponderande *symmetriska balanserade inkompletta blockdesignen* (SBIBD) är balanserad. För detta presenterar vi nya nödvändiga villkor. Speciellt visar vi att den residuala designen till den korresponderande SBIBDen måste vara kvasi-symmetrisk och ger nödvändiga och tillräckliga villkor för dess blockskärningstal. Vi adresserar också frågan om när den inre designen är balanserad med avseende på alla SBIBDens block. Vi visar att en sådan SBIBD måste ha den egenskap som kallas kvasi-3 och svarar på existensfrågan för alla kända klasser av sådana designer.

Eftersom snittbalanserade triple arrays verkar vara väldigt sällsynta är det naturligt att fråga om det finns andra familjer av rad-kolumn designer som har denna egenskap. I artikel II ger vi nödvändiga och tillräckliga villkor för att en balanced grid ska vara snittbalanserad och visar att alla designer i en oändlig familj av binära pseudo-Youden squares är snittbalanserade.

Existensfrågan för triple arrays är öppen fråga. Det finns en konstruktionsmetod för en oändlig men speciell familj kallad Paley triple arrays och så finns det en allmän metod för vilken ett steg är obevisat. I artikel III undersöker vi en tredje konstruktionsmetod som utgår från Youden squares. Denna metod föreslogs i litteraturen för länge sedan men blev motbevisad med hjälp av ett motexempel. Vi visar bland annat att Youden squares från projektiva plan aldrig kan ge en triple array med denna metod, men att det för varje triple array som korresponderar till ett biplan, så finns det en lämplig Youden square för vilken metoden fungerar. Vidare konstruerar vi familjen av Paley triple arrays med denna metod.

Inom matematisk fysik arbetar vi med solitoner som man i naturen kan få se som självförstärkande vågor vilka beter sig som partiklar. Inom matematiken är de lösningar till vissa ickelinjära differentialekvationer. I artikel IV studerar vi det tvådimensionella Toda-gittret för vilken vi konstruerar en familj av lösningar och även explicita lösningsformler.

Acknowledgements

First of all, I would like to express my gratitude to my supervisor Professor Cornelia Schiebold. Thank you for always being both inspiring and patient with this old student. I would also like to thank my assistant supervisors: Egmont Porten for his support, Lars-Daniel Öhman with whom I hope to do many papers in the future and Frank Wikström who was my assistant supervisor in the beginning of my doctoral studies.

I thank Jörgen Boo who was very important for me in my early studies and also my co-author Pia Heidtmann who introduced me to design theory and was my Master Thesis supervisor.

To Stefan Borell, Andreas Lind, Sam Lodin, Per Åhag, Abtin Daghighi and Peter Glans. At a small university like Mid Sweden University, your friends and colleagues become very important. For all discussions, help, inspiration and joyful moments, but most of all for your friendship, I thank you.

Finally, thanks Elle, Oskar and Liu for your patience with this young man who so often loses himself in his work.

Contents

List of Papers

This thesis is based on the following papers, herein referred by their Roman numerals:

I T. Nilson and P. Heidtmann. *Inner balance of symmetric designs*. Designs, Codes and Cryptograhy, DOI 10.1007/s10623-012-9730-2.

II T. Nilson. *Pseudo-Youden designs balanced for intersection*. J. Statist. Plann. Inference, 141, (2011), pp. 2030–2034.

III T. Nilson and L.-D. Öhman. *Triple arrays and Youden squares,* manuscript.

IV T. Nilson and C. Schiebold. *On the noncommutative two-dimensional Toda lattice,* manuscript.

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List of Figures

List of Tables

Notation

$\mathrm{Aut}(\mathcal{D})$	The full automorphism group of a design. $\mathcal{D}$
BIBD	Balanced incomplete block design.
D_B	The derived design of $\mathcal{D}$ with respect to the block. B.
D^B	The residual design of $\mathcal{D}$ with respect to the block. B.
D'	The complementary design of $\mathcal{D}$.
$D_\star$	The inner design of $\mathcal{D}$.
$\mathcal{D}^\tau$	The dual design of $\mathcal{D}$.
$GF(q)$	A finite field of order q.
I_v	The identity matrix of order v.
$J_{v,b}$	A $v \times b$ matrix in which all entries are 1.
KdV	Korteweg–de Vries equation.
KP	Kadomtsev–Petviashvili equation.
$PG(n,q)$	A n-dimensional projective space over $GF(q)$
$PG_d(n,q)$	An incidence structure formed by the points and the d-dimensional subspaces of $PG(n,q)$.
SBIBD	Symmetric balanced incomplete block design.

Chapter 1

Entrance

Since this thesis treats two different areas, one could ask if they have something in common. Well, we will see that both areas were started up about the same time, around the year 1840. Another common denominator is *balance*. In Paper I and II, we search for an *inner balance* of designs. In Paper IV we study solitons which are said to have *perfect balance*, as the tendencies to disperse and to break cancel each other out.

1.1 Outline of the thesis

Part I is devoted to design theory. In Chapter 2 we give quite a general introduction to *balanced incomplete block designs*, although the material is selected to also give a first background for Paper I-III. In Chapter 3 we introduce row-column designs like Youden squares and triple arrays, and look at construction methods for the latter ones. In Chapter 4 we declare the research questions for Paper I-III, give additional background for these and what results we have achieved.

In Part II we treat soliton theory. After a short historical introduction of the general kind we use most of Chapter 5 to introduce the operator method, and to exemplify it on the Korteweg–de Vries equation. Finally, we give a short summary of Paper IV.

Part I

Designs

Chapter 2

Block designs

"Never underestimate a theorem that counts something!"
John B. Fraleigh. A First Course in Abstract Algebra.

2.1 Design theory – a part of combinatorics

Combinatorial design theory is a branch of mathematics that deals with existence, construction and properties of systems of finite sets whose arrangements satisfy certain concepts like balance or symmetry.

In the past, these structures were studied principally for their aesthetic appeal, but in the early 1900s they came to great practical use when the theory of statistical experiments was developed. When statisticians like R. Fisher in the 1920s laid the foundations of the theory, it was intimately linked to such applications. Then, in the 1930s, R. C. Bose and his colleagues took it further by developing deep connections with number theory, algebra and finite geometry. Nowadays, design theory is a field in its own right even if it is still closely connected to applications in statistics. But, like most areas of combinatorics, it has been growing fast in the last 30 years since the need for discrete structures has increased in the computer age.

The study of block designs can be traced back to 1835 when Plücker [29] in a study of algebraic curves encountered a design which we now call a Steiner triple system, but we will take our starting point a few years later in recreational mathematics. In the early 1800s the publication *Lady's and Gentlemen's Diary* devoted several columns to mathematical problems. One such example is the following price question published in 1844 by W. Woolhouse:

> *"Determine the number of combinations that can be made out of v symbols, each combination having k symbols, with this limitation, that no combination of t symbols which may appear in any one of them, may appear in any other."*

This problem turned out to be very hard and it is still open, (the existence question for Steiner systems[1]), but T. P. Kirkman [22] solved it completely in the special case of $t = 2$ and $k = 3$, in which he gave necessary and sufficient conditions on v. As a byproduct of this work he then published a more manageable problem in the Diary known as *Kirkman's schoolgirl problem*:

[1]A Steiner system is a $(v, k, 1)$-BIBD. J. Steiner asked about the existence of $(v, 3, 1)$-BIBDs in 1853, unaware of Kirkman's work. As Steiner is more widely known, these systems were named in his honor.

> *"Fifteen young ladies in a school walk out three abreast for seven days in succesion: it is required to arrange them daily so that no two shall walk twice abreast."*

If the young ladies are numbered $1, 2, \ldots, 15$, the following arrangement is a solution. Each pair of girls walk together exactly once during the week.

Monday:	$\{1,6,11\},\{2,7,12\},\{3,8,13\},\{4,9,14\},\{5,10,15\}$
Tuesday:	$\{1,2,5\},\{3,4,7\},\{8,9,12\},\{10,11,14\},\{13,15,6\}$
Wednesday:	$\{2,3,6\},\{4,5,8\},\{9,10,13\},\{11,12,15\},\{14,1,7\}$
Thursday:	$\{5,6,9\},\{7,8,11\},\{12,13,1\},\{14,15,3\},\{2,4,10\}$
Friday:	$\{3,5,11\},\{4,6,12\},\{7,9,15\},\{8,10,1\},\{13,14,2\}$
Saturday:	$\{5,7,13\},\{6,8,14\},\{9,11,2\},\{10,12,3\},\{15,1,4\}$
Sunday:	$\{11,13,4\},\{12,14,5\},\{15,2,8\},\{1,3,9\},\{6,7,10\}$

Figure 2.1: A solution of Kirkman's schoolgirl problem for 15 girls.

A first solution was published by A. Cayley, but the problem and especially its generalizations have continued to attract attention.

2.2 Balanced incomplete block designs

In this thesis we deal with properties, existence and construction of these designs. About their usefulness in statistical experiments we confine ourselves to mention that statisticians for this consider several types of *optimality*, and to quote Bailey and Cameron [5] who wrote: *"Kiefer's Theorem asserts that balanced incomplete block designs, if they exist, are optimal in any reasonable sense"*.

2.2.1 Basic definitions and properties

The solution of Kirkman's schoolgirl problem in Figure 2.1 is a special example of a *balanced incomplete block design*.

Definition 2.1. *A* combinatorial design *$\mathcal{D}$ is a pair $(X, \mathcal{B})$, where X is a set of v elements called points, and $\mathcal{B}$ is a collection of b subsets of X called blocks.*

1. *If there exist positive integers k, r such that each block contains exactly k points and each point occur in exactly r blocks, then $\mathcal{D}$ is called a* block design.
2. *A block design is called* complete *if $k = v$ and* incomplete *if $k < v$.*
3. *A block design is* balanced *if there exists a positive integer λ such that any 2-subset of X occurs in exactly λ of the blocks. In this case we call λ the* index *of the design.*

We denote a balanced incomplete block design by BIBD and write the parameters (v, b, r, k, λ) or just (v, k, λ), since b and r then will be obtainable. The order *of a BIBD is the non-negative integer $r - \lambda$. A BIBD where $v = b$ is said to be* symmetric *(or* square*) and is denoted by SBIBD.*

In this thesis we mainly use the notion BIBDs but they are also called 2-designs according to the following generalization.

Definition 2.2. *Let v, k, λ and t be positive integers such that $v > k \geq t$. A $t-(v, k, \lambda)$-design is a design $(X, \mathcal{B})$ such that the following properties are satisfied:*

1. *$|X| = v$,*
2. *each block contains exactly k points, and*
3. *every set of t distinct points is contained in exactly λ blocks.*

The general term t-design is used to indicate any $t-(v, k, \lambda)$-design.

A block design is often given by a list of its blocks, like the $(15, 35, 7, 3, 1)$-BIBD in Figure 2.1, but the incidence structure can also be represented by a matrix.

Definition 2.3. *The* incidence matrix *of a block design $(X, \mathcal{B})$ with parameters (v, b, r, k) is a $v \times b$ matrix $A = [a_{ij}]$ in which $a_{ij} = 1$ when the ith element of X occurs in the jth block of $\mathcal{B}$, and $a_{ij} = 0$ otherwise.*

Example 2.4. *A $(6, 10, 5, 3, 2)$-BIBD $\mathcal{D} = (X, \mathcal{B})$, represented by an incidence matrix in which we have labeled the rows and columns by the elements of $X = \{1, 2, \ldots, 6\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_{10}\}$ respectively.*

$A =$

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}
1	1	1	1	1	1	0	0	0	0	0
2	1	1	0	0	0	1	1	1	0	0
3	0	0	1	1	0	1	1	0	1	0
4	0	0	1	0	1	1	0	1	0	1
5	1	0	0	1	0	0	0	1	1	1
6	0	1	0	0	1	0	1	0	1	1

We will now look at some fundamental properties for BIBDs.

Proposition 2.5. *Let $\mathcal{D}$ be a block design with parameters (v, b, r, k), then*

1. *$vr = bk$.*
2. *If $\mathcal{D}$ is balanced with index λ, then $\lambda(v-1) = r(k-1)$.*

That Proposition 2.5 is true can be understood by "double counting" in the matrix A in Example 2.4. Further inspection of A also makes the following result believable.

Theorem 2.6. *A $(0, 1)$-matrix A with v rows and b columns is the incidence matrix of a (v, b, r, k, λ)-BIBD if and only if the following conditions are satisfied.*

1. *$J_v A = k J_{v,b}$, where $2 \leq k < v$;*
2. *$AA^T = (r - \lambda) I_v + \lambda J_v$.*

where I is the identity matrix and J is the all-one matrix.

In condition (1) of Theorem 2.6 its checked that the block size is constant. This because there is a related type of designs called *pairwise balanced designs*, which satisfy (2), but have two or more block sizes. The use of incidence matrices open for notions and tools from linear algebra as in the following examples.

Lemma 2.7. *Let A be the incidence matrix of a (v, b, r, k, λ)-BIBD. Then* $\det(AA^T) = (r - \lambda)^{v-1} rk$.

Proposition 2.5 and $k < v$ gives that $\lambda = r(k-1)/(v-1) < r$, so AA^T is non-singular. Further, for a SBIBD we have that A is square and $r = k$, which together with $\det(A) = \det(A^T)$ and $\det(AA^T) = \det(A)\det(A^T) = (\det(A))^2$ gives the following corollary.

Corollary 2.8. *Let $\mathcal{D}$ be a (v, k, λ)-SBIBD and A be an incidence matrix of $\mathcal{D}$. Then A is non-singular and* $\det(A) = (k - \lambda)^{(v-1)/2} k$.

The following simple but useful result by R. Fisher can be proved by observing that $v = \text{rank}(AA^T) \leq \text{rank}(A) \leq \min\{v, b\}$.

Theorem 2.9. *(Fisher's inequality) For any (v, b, r, k, λ)-BIBD, the number of points does not exceed the number of blocks, i.e., $v \leq b$.*

From a given design it is possible to form new designs.

Definition 2.10. *Let $\mathcal{D}$ be a combinatorial design. The* dual design *of $\mathcal{D}$, denoted $\mathcal{D}^\tau$ is obtained by interchanging the roles of blocks and points.*

Fisher's inequality gives that a necessary condition for the dual design of a BIBD $\mathcal{D}$ to also be balanced is that $\mathcal{D}$ is a SBIBD, and this condition also turns out to be sufficient.

Theorem 2.11. *Let $\mathcal{D}$ be a SBIBD. Then the dual design $\mathcal{D}^\tau$ is also a SBIBD.*

Proof. Let A denote the incidence matrix of a (v, k, λ)-SBIBD $\mathcal{D}$. Then $\mathcal{D}^\tau$ is a block-design with parameters (v, v, k, k) and incidence matrix A^T. By Theorem 2.6 $\mathcal{D}^\tau$ is balanced if we can show that $A^T A = (k - \mu)I_v + \mu J_v$ for some positive integer μ. We check (2) of Theorem 2.6. Note that the square matrices here are all of order v and that A is invertible by Corollary 2.8.

$$\begin{aligned} AA^T &= (k - \lambda)I + \lambda J \\ A^{-1}(AA^T)A &= A^{-1}\left((k - \lambda)I + \lambda J\right)A \\ A^T A &= (k - \lambda)I + \lambda A^{-1} J A, \end{aligned}$$

and as the replication number is equal to the block size k we here have $JA = AJ$ which gives the result. □

There is an important corollary of this result concerning block intersections.

Corollary 2.12. *Suppose B_i and B_j are two distinct blocks in a (v, k, λ)-SBIBD. Then $|B_i \cap B_j| = \lambda$.*

We conlude this first part of the introduction by defining the complementary design.

Definition 2.13. *Let $\mathcal{D} = (X, \mathcal{B})$ be a block design. The* complementary design *of $\mathcal{D}$, denoted $\mathcal{D}'$, has point set X and the blocks are the sets $X \setminus B_i$ for $B_i \in \mathcal{B}$.*

Proposition 2.14. *Let $\mathcal{D}$ be a (v, b, r, k, λ)-BIBD. Then $\mathcal{D}'$ is a $(v, b, b - r, v - k, b - 2r + \lambda)$-BIBD, provided that $b - 2r + \lambda > 0$.*

The condition $b - 2r + \lambda > 0$ is not very restrictive as it only excludes BIBDs with $v = k + 1$.

2.2.2 When are two designs the same?

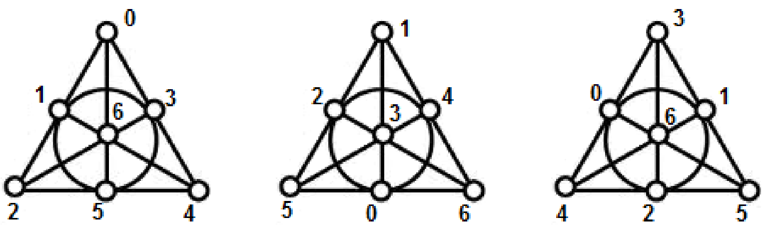
If two designs with the same parameters have the "same structure" we say that they are isomorphic. For example, there exist 80 non-isomorphic $(15, 35, 7, 3, 1)$-BIBDs , and among these 80, only seven solve Kirkman's schoolgirl problem (cf.[14] p. 66).

Definition 2.15. *Suppose* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *are two combinatorial designs with* $|X| = |Y|$. *Then* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *are* isomorphic *if there exists a bijection* $f : X \to Y$ *such that*

$$[\{f(x) : x \in A\} : A \in \mathcal{A}] = \mathcal{B}.$$

In other words, if we rename every point $x \in X$ *by* $f(x)$, *then the collection of blocks* $\mathcal{A}$ *is transformed into* $\mathcal{B}$. *The bijection* f *is called an* isomorphism.

An isomorphism from a BIBD $D = (X, \mathcal{B})$ to itself is called an *automorphism* of $\mathcal{D}$. All automorphisms of $\mathcal{D}$ form the *full automorphism group* of $\mathcal{D}$ denoted by $\text{Aut}(\mathcal{D})$, which is a subgroup of the symmetric group S_X, the group of all $|X|!$ permutations of the set X.



(a) $\mathcal{D}_0 = (X, \mathcal{B})$ (b) $\mathcal{D}_1 = (X, \mathcal{A})$ (c) $\mathcal{D}_2 = (X, \mathcal{B})$

Figure 2.2: Three Fano planes.

Example 2.16. *A* $(7, 3, 1)$-*SBIBD is sometimes given a graphic representation called the* Fano plane [2] *which can be seen in Figure 2.2. The three SBIBDs given there all have point set* $X = \{0, 1, 2, 3, 4, 5, 6\}$ *and their block sets are listed here below:*

$\mathcal{D}_0 = (X, \mathcal{B})$, $\mathcal{B} = \{\{0, 1, 2\}, \{0, 3, 4\}, \{0, 5, 6\}, \{1, 3, 5\}, \{1, 4, 6\}, \{2, 3, 6\}, \{2, 4, 5\}\}$,

$\mathcal{D}_1 = (X, \mathcal{A})$, $\mathcal{A} = \{\{1, 2, 5\}, \{1, 4, 6\}, \{1, 0, 3\}, \{2, 4, 0\}, \{2, 6, 3\}, \{5, 4, 3\}, \{5, 6, 0\}\}$,

$\mathcal{D}_2 = (X, \mathcal{B})$, $\mathcal{B} = \{\{3, 0, 4\}, \{3, 1, 5\}, \{3, 2, 6\}, \{0, 1, 2\}, \{0, 5, 6\}, \{4, 1, 6\}, \{4, 5, 2\}\}$.

We can define a permutation $\sigma = (0\ 1\ 2\ 5)(3\ 4\ 6)$ *on* X *that transforms the block set* $\mathcal{B}$ *of* $\mathcal{D}_0$ *to the block set* $\mathcal{A}$ *of* $\mathcal{D}_1$, *so* σ *is an isomorphism and* $\mathcal{D}_0$ *and* $\mathcal{D}_1$ *are isomorphic, but note that* $\mathcal{A} \neq \mathcal{B}$ *so* σ *is not an automorphism. On the other hand, we can define the permutation* $\tau = (0\ 3\ 1)(2\ 4\ 5)(6)$ *on* X *that transforms the block set of* $\mathcal{D}_0$ *to the block set of* $\mathcal{D}_2$. *In this case, the block sets are equal so* τ *is an automorphism.*

It can be mentioned that the $(7, 3, 1)$-SBIBD $\mathcal{D}$ is unique up to isomorphism and that $\text{Aut}(\mathcal{D})$ is the *projective special linear group* $PSL(2, 7)$ of order 168.

2.3 Symmetric designs and their subdesigns

The term "symmetric" is inherited from the early days of the subject and the incidence matrix of a SBIBD is seldom symmetric. Some authors use the term square

[2] The Fano plane is named after Gino Fano (1871–1952) who worked in projective and algebraic geometry. A $(v, k, 1)$-SBIBD is a projective plane of order $k - 1$ as we will see in Section 2.3.2

instead but symmetric is still the most commonly used.

2.3.1 Subdesigns and block intersections

Subdesigns are defined as follows.

Definition 2.17. *A block design* $(Y, \mathcal{C})$ *is a* subdesign *of a block design* $(X, \mathcal{B})$ *if and only if* $Y \subseteq X$ *and* $\mathcal{C} \subseteq \mathcal{B}$*. The subdesign is* proper *if* $Y \subset X$*.*

Given a SBIBD we want to construct subdesigns with nice properties, and the following two kinds of substructures are of special interest.

Definition 2.18. *Let* $\mathcal{D} = (X, \mathcal{B})$ *be an SBIBD and let* $B_0 \in \mathcal{B}$*. The* derived design *of* $\mathcal{D}$ *with respect to* B_0*, denoted* $\mathcal{D}_{B_0}$*, has point set* B_0 *and the blocks are the sets* $B_i \cap B_0$*, for* $B_i \in \mathcal{B} \setminus \{B_0\}$*.*

Proposition 2.19. *Let* $\mathcal{D}$ *be a* (v, k, λ)*-SBIBD and let* B *be a block of* $\mathcal{D}$*. Then* $\mathcal{D}_B$ *is a* $(k, v-1, k-1, \lambda, \lambda-1)$*-BIBD, provided that* $\lambda \geq 2$*.*

Definition 2.20. *Let* $\mathcal{D} = (X, \mathcal{B})$ *be an SBIBD and let* $B_0 \in \mathcal{B}$*. The* residual design *of* $\mathcal{D}$ *with respect to* B_0*, denoted* $\mathcal{D}^{B_0}$*, has point set* $X \setminus B_0$ *and the blocks are the sets* $B_i \setminus B_0$*, for* $B_i \in \mathcal{B} \setminus \{B_0\}$*.*

Proposition 2.21. *Let* $\mathcal{D}$ *be a* (v, k, λ)*-SBIBD and let* B *be a block of* $\mathcal{D}$*. Then* $\mathcal{D}^B$ *is a* $(v-k, v-1, k, k-\lambda, \lambda)$*-BIBD.*

Example 2.22. *An incidence matrix of an* $(11, 5, 2)$*-SBIBD where the entries in column* 0*, (block* B_0*), have been emphasized. Rows* $0, 1, \ldots, 4$*, and columns* $1, 2, \ldots, 10$*, form the incidence matrix of the derived design with respect to the block* B_0*, which is a* $(5, 10, 1)$*-BIBD. Rows* $5, 6, \ldots, 10$*, and columns* $1, 2, \ldots, 10$*, form the incidence matrix of the residual design with respect to* B_0*, which is a* $(6, 3, 2)$*-BIBD.*

	0	1	2	3	4	5	6	7	8	9	10
0	**1**	0	0	1	0	0	0	1	1	1	0
1	**1**	0	1	0	0	1	0	0	0	1	1
2	**1**	1	0	1	0	0	1	0	0	0	1
3	**1**	1	1	0	1	0	0	1	0	0	0
4	**1**	0	0	0	1	1	1	0	1	0	0
5	0	1	0	0	1	0	0	0	1	1	1
6	0	1	1	1	0	1	0	0	1	0	0
7	0	0	1	1	1	0	1	0	0	1	0
8	0	0	0	1	1	1	0	1	0	0	1
9	0	1	0	0	0	1	1	1	0	1	0
10	0	0	1	0	0	0	1	1	1	0	1

Any (v, b, r, k, λ)-BIBD $\mathcal{D}$ with $k = r - \lambda$ is called a *quasi-residual* design as it has the parameters to be a residual design of some SBIBD. If $\mathcal{D}$ is a residual of a $(v + r, r, \lambda)$-SBIBD, then it is said to be *embeddable*. Else, $\mathcal{D}$ is said to be *non-embeddable*.

Remark 2.23. *Derived and residual designs can also be defined with respect to a point. Starting with a BIBD* $\mathcal{D}$*, we define the* derived design of $\mathcal{D}$ with respect to a point x*, denoted* $\mathcal{D}_x$*, to be what remains when we remove the point* x *together with all blocks not incident with* x*. The* residual design of $\mathcal{D}$ with respect to x *is denoted* $\mathcal{D}^x$ *and consists of the blocks which are not incident with* x*.*

The theory of block intersections is an area where we have limited knowledge. In Corollary 2.12 we established that any pair of distinct blocks in a (v, k, λ)-SBIBD intersect in λ points, but the picture becomes more unclear when we consider BIBDs with $v < b$.

Definition 2.24. *Suppose $\mathcal{D}$ is a block design with blocks $B_0, B_1, \ldots, B_{b-1}$. The distinct cardinalities $|B_i \cap B_j|, i \neq j$, are called the* intersection numbers *of $\mathcal{D}$.*

Using Fisher's inequality we can deduce that BIBDs in general have more than one intersection number.

Proposition 2.25. *A (v, b, r, k, λ)-BIBD with $v < b$ has at least two intersection numbers.*

The $(6, 10, 5, 3, 2)$-BIBD in Example 2.4 has exactly two intersection numbers, $|B_1 \cap B_2| = 2$ and $|B_2 \cap B_3| = 1$. This design belongs to a class of designs which are relatively well-studied when block intersections are considered.

Definition 2.26. *A BIBD with exactly two intersection numbers is called a* quasi-symmetric *design.*

There are many results for special cases of such designs, but the classification of quasi-symmetric designs is still an open problem. Here, we give one example of a small, but general result which can be very useful.

Proposition 2.27. (*[40], [17]*). *If x and y, $x < y$, are the intersection numbers of a quasi-symmetric (v, b, r, k, λ)-BIBD, then $y - x$ divides both $k - x$ and $r - \lambda$.*

We now turn back to SBIBDs where we will define a property and an extremal class of these designs according to block intersections. One can say that a SBIBD is *regular* as any block intersects with itself in k points and *doubly regular* as any two distinct blocks intersect in λ points. Thus, we could say that a SBIBD is *triply regular* if any three distinct blocks intersect in the same number of points. However, this only happens in trivial SBIBDs with parameters $(v, v-1, v-2)$, and never in an SBIBD $\mathcal{D}$ with $2 \leq k \leq v-2$. This because the dual design $\mathcal{D}^\tau$ would be a 3-design, which is impossible by the following well-known result.

Lemma 2.28. *A $t - (v, k, \lambda)$ design $\mathcal{D} = (X, \mathcal{B})$ with $t \geq 3$ and $k \leq v - 2$ cannot be a symmetric design.*

Proof. Let $x \in X$ and let Y be a $t-1$ subset of $X \setminus \{x\}$. Then $Y \cup \{x\}$ is a t-subset of X. Therefore, there are exactly λ blocks $B \in \mathcal{B}$ that contain x and contain Y, which means that $\mathcal{D}_x$, the derived design with respect to the point x, is a $(t-1) - (v-1, k-1, \lambda)$ design.

If $t \geq 3$, then $\mathcal{D}_x$ is a 2-design with $v - 1$ points. The replication number of $\mathcal{D}$ is k so $\mathcal{D}_x$ has $k \leq v - 2$ blocks, but this cannot be by Fisher's inequality 2.9. □

Therefore, a SBIBD where the cardinality of the intersection of any three distinct blocks just takes on one of two values is called *nearly triply regular*, or more commonly *quasi*-3.

Definition 2.29. *An SBIBD $\mathcal{D}$ is said to be* quasi-3 *if there exist integers x and y, called* triple intersection numbers, *such that $|A \cap B \cap C| \in \{x, y\}$ for any three distinct blocks A, B, and C of $\mathcal{D}$.*

The $(11, 5, 2)$-SBIBD in Example 2.22 is quasi-3 with triple intersection numbers 0 and 1, and we note that these are also the intersection numbers of the derived design with respect to the block B_0.

Proposition 2.30. *(cf. [19], p. 263) A non-trivial SBIBD $\mathcal{D} = (X, \mathcal{B})$ is quasi-3 if and only if the derived design $\mathcal{D}_B$ is quasi-symmetric for every $B \in \mathcal{B}$.*

Sometimes we emphasize that a SBIBD is *quasi-3 for blocks*. This because the dual definition *quasi-3 for points* is also used, and this was how quasi-3 designs were first introduced by Cameron [11].

2.3.2 Affine and projective planes

Many designs come from finite geometry and in this section we give an axiomatic descrition of the two main kinds of finite plane geometry, affine and projective planes, and how they are related to designs.

In Euclidean plane geometry we study the incidence structures formed by points and lines in a plane. If there is a unique line trough any two distinct points, and for any point not on a given line, there is a unigue line on the point that is parallel to (i.e., disjoint from) the given line, then an incidence structure with these properties is called an *affine plane*.

Definition 2.31. *An* affine plane *is a pair $(X, \mathcal{L})$, where X is a non-empty set of elements called* points *and $\mathcal{L}$ is a family of subsets of X called* lines, *that satisfy the following axioms:*

(A1) Any two distinct points lie on a unique line.

(A2) For any line L and any point $x \neq L$, there is a unique line M that contains x and is disjoint from L.

(A3) There exists a triangle, *i.e., a set of three points not on a common line.*

Two lines with empty intersection are said to be *parallel*, and this relation between lines is called *parallelism*. It is an equivalence relation on the lines of an affine plane and the equivalence classes are called *parallel classes*.

From the axioms of an affine plane it is possible to deduce the following properties.

Theorem 2.32. *(cf. [19], p. 63) For any finite affine plane $\mathcal{A}$ there is a positive integer $n \geq 2$ such that every line of $\mathcal{A}$ consists of exactly n points, every point lies on exactly $n+1$ lines, and $\mathcal{A}$ has exactly n^2 points, $n^2 + n$ lines and $n+1$ parallel classes.*

We say that such an affine plane is of *order* n. The smallest affine plane consists of six lines which are all the 2-subsets of the point set of four points. The affine plane of order 3 is of some historical interest as it was the structure presented by Plücker [29] back in 1835.

Affine planes can be constructed as follows. Let X be a 2-dimensional vector space over $GF(q)$, the finite field of order q. For any $m, b \in GF(q)$ we call the set $\{(x, y) \in X : y = mx + b\}$ a line with slope m. For any $a \in GF(q)$, we call the set $\{(x, y) \in X : x = a\}$ a line with infinite slope. If $\mathcal{L}$ is the set of all lines, then $(X, \mathcal{L})$ is an affine plane of order q which we denote $AG(2, q)$.

Theorem 2.33. *(cf. [19], p. 63) For any prime power q, there exists an affine plane of order q.*

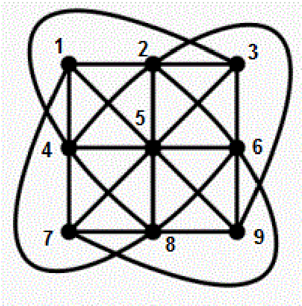


Figure 2.3: An affine plane of order 3. The three lines $\{4, 2, 9\}, \{7, 5, 3\}$ and $\{1, 8, 6\}$ form one of the four parallel classes.

By the properties given in Theorem 2.32 together with axiom (A1) we understand that an affine plane is also a BIBD.

Proposition 2.34. *(cf. [19], p. 63) An affine plane of order n is an $(n^2, n, 1)$-BIBD, and conversely, for $n \geq 2$, any $(n^2, n, 1)$-BIBD is an affine plane of order n.*

Affine planes have parallel lines and give us BIBDs with the corresponding property. In a projective plane, by contrast, any two lines intersect so parallel lines do not exist.

Definition 2.35. *A* projective plane *is a pair $(X, \mathcal{L})$ where X is a non-empty set of elements called* points *and $\mathcal{L}$ is a family of subsets of X called* lines, *that satisfy the following axioms:*

(P1) Any two distinct points lie on a unique line.

(P2) Any two lines have a non-empty intersection.

(P3) There exists an quadrangle, *i.e., a set of four points, no three of which lie on a common line.*

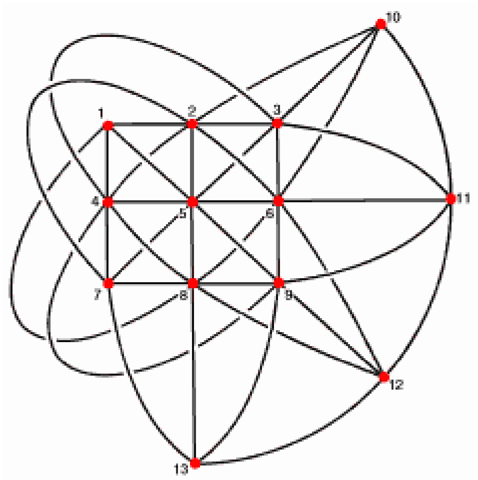


Figure 2.4: A projective plane of order 3.

An affine plane can be extended to a projective plane by adding a "line of infinity" consisting of its parallel classes.

Theorem 2.36. *(cf. [19], p. 72) Let $\mathcal{A} = (X, \mathcal{L})$ be an affine plane. Let Π be the set of parallel classes in $\mathcal{A}$. Put $X' = X \cup \Pi$. For each line L in $\mathcal{A}$, put $L' = L \cup \pi$ where π is the parallel class containing L. Finally put $\mathcal{L}' = \{L' : L \in \mathcal{L}\} \cup \{\Pi\}$, Then $\mathcal{P} = (X', \mathcal{L}')$ is a projective plane.*

The converse result is also true. Remove any line together with its points from a projective plane and what remains is an affine plane.

Theorem 2.37. *(cf. [19], p.73) Let $\mathcal{P} = (X, \mathcal{L})$ be a projective plane and let L be a line of $\mathcal{P}$. Let $X' = X \setminus L$ and $\mathcal{L}' = \mathcal{L} \setminus \{L\}$. Then $\mathcal{A} = (X', \mathcal{L}')$ is an affine plane.*

It follows that a projective plane is a SBIBD. We say that a projective plane $\mathcal{P}$ is of *order* n if each line of $\mathcal{P}$ has cardinality $n + 1$.

Theorem 2.38. *(cf. [19], p. 73) A projective plane of order n is a $(n^2 + n + 1, n + 1, 1)$-SBIBD, and conversely, any symmetric $(n^2 + n + 1, n + 1, 1)$-SBIBD with $n \geq 2$ is a projective plane of order n.*

We see that there is a projective plane of order n if and only if there is a affine plane of order n.

Theorem 2.39. *For any prime power $q \geq 2$, there exists a projective plane of order q.*

2.3.3 Difference sets

We will now look at an important construction method for SBIBDs.

Let $\mathcal{D} = (X, \mathcal{B})$ be a SBIBD. Suppose there is a group $G \subseteq \mathrm{Aut}(\mathcal{D})$ acting on a block $B \in \mathcal{B}$ such that the orbit of B is all of $\mathcal{B}$, then we immediately get $\mathcal{D}$ back. Such a block B is called a *base block* or a *starter block*, but a moment of thought gives that all blocks in the orbit would do as base blocks.

Now we want to start in the other end and use this to construct a SBIBD. We need to find a suitable pair of a subset D and a group G such that the action of G on D gives a SBIBD. We do that by taking the group G as the point set of the design and consequently the base block D will be a subset of G. Such base blocks are called *difference sets*.

Definition 2.40. *Let G be a finite group of order v. A k-subset D of G is called a (v, k, λ)-* difference set *if the multiset*

$$\{xy^{-1} : x, y \in D, x \neq y\}$$

contains exactly λ copies of every non-identity element of G. Difference sets in abelian groups are called abelian difference sets *and difference sets in cyclic groups are called* cyclic difference sets.

Note that the identity $\lambda(v - 1) = k(k - 1)$ holds for difference sets as well as for SBIBDs.

Example 2.41. *Let $G = (\mathbb{Z}_7, +)$. Then $D = \{0, 3, 5, 6\}$ is a $(7, 4, 2)$-difference set in G. Every non-zero difference occurs exactly two times as a difference between the elements of D.*

It is often possible to find difference sets in groups of type $(\mathbb{Z}_p, +)$, but there are also other kinds.

Example 2.42. *There is no* $(16, 6, 2)$*-difference set in* $(\mathbb{Z}_{16}, +)$*, yet it is possible to find such difference sets in other groups of this order. One non-cyclic but abelian example is* $D = \{(0, 2), (1, 0), (1, 1), (1, 3), (2, 2), (3, 2)\}$ *in the group* $(\mathbb{Z}_4 \times \mathbb{Z}_4, +)$*.*

A non-abelian example is the group $\mathbb{H} \times \mathbb{Z}_2$ *where* $\mathbb{H} = \{\pm 1, \pm i, \pm j, \pm k\}$ *is the quarternion group in which the generators satisfy* $i^2 = j^2 = k^2 = ijk = -1$*. The set* $D = \{(1, 0), (i, 0), (j, 0), (k, 0), (1, 1), (-1, 1)\}$ *is a non-abelian* $(16, 6, 2)$*-difference set in* $\mathbb{H} \times \mathbb{Z}_2$*.*

Paley difference sets are probably the best known class of difference sets. They are defined in $GF(q)$, the finite field of order q, as given in the following theorem.

Theorem 2.43. *(cf. [25], p. 375) Let* $q = 4n - 1$ *be a prime power. Then the set* Q *of non-zero quadratic residues in* $GF(q)$ *is a* $(4n - 1, 2n - 1, n - 1)$*-difference set in the additive group of* $GF(q)$*.*

Now we turn to the construction of SBIBDs and start by giving a name to the structure obtained when a group acts on one of its subsets.

Definition 2.44. *Let* X *be a subset of a finite group* G*. For any* $g \in G$*, define*

$$Xg = \{xg : x \in X\}.$$

We call any set Xg *a* translate *of* X *and define the* development *of* X*, denoted* $\mathrm{Dev}(X)$*, to be the collection of all translates of* X*.*

The following result gives us a large class of groups where difference sets can be used.

Theorem 2.45. *(cf. [42], p. 43) Let* D *be a* (v, k, λ)*-difference set in an abelian group* $(G, +)$*. Then* $(G, \mathrm{Dev}(D))$ *is a* (v, k, λ)*-SBIBD.*

As the additive group of $GF(q)$ is abelian, Theorems 2.45 and 2.43 give that for each prime power $q = 4n - 1$ where $n > 1$, there is a $(4n - 1, 2n - 1, n - 1)$-SBIBD. These designs lie in the class of Hadamard 2-designs which consists of all $(4n - 1, 2n - 1, n - 1)$-SBIBDs. [3]

To construct a Hadamard 2-design from a Paley difference set becomes very easy when q is a prime.

Example 2.46. *The set of non-zero quadratic residues in* $GF(11)$ *is* $Q = \{1, 3, 4, 5, 9\}$ *and this is a* $(11, 5, 2)$*-difference set. By Theorem 2.45 we know that* $((GF(11), +), \mathrm{Dev}(Q))$ *is a* $(11, 5, 2)$*-SBIBD. It is given here below, where we have written the blocks as columns.*

1	2	3	4	5	6	7	8	9	10	0
3	4	5	6	7	8	9	10	0	1	2
4	5	6	7	8	9	10	0	1	2	3
5	6	7	8	9	10	0	1	2	3	4
9	10	0	1	2	3	4	5	6	7	8

In order to give a more general description of when a SBIBD can be constructed from a difference set, we need the following property for group actions.

Definition 2.47. *An action of a group* G *on a set* X *is said to be* sharply transitive *if for any* $x, y \in X$ *there is a unique* $\sigma \in G$ *such that* $\sigma x = y$*.*

[3] A $(4n-1, 2n-1, n-1)$ Hadamard 2-design, $n > 1$ exists if and only if a corresponding Hadamard matrix exists. That is a square (± 1)-matrix H of order $m = 4n$ satisfying $HH^T = mI_m$.

That a group G acts sharply transitive on a difference set D means that there will be a single orbit which is necessary for the following generalization of Theorem 2.45 for arbitrary finite groups.

Theorem 2.48. *(cf. [19], p. 295) A SBIBD $\mathcal{D}$ can be obtained as the development of a difference set in a group G if and only if $\mathrm{Aut}(\mathcal{D})$ has a sharply transitive subgroup isomorphic to G.*

One example of an application of this theoreom is if we are given that there exists a $(16, 6, 2)$-SBIBD $\mathcal{D}$ and that $(\mathbb{Z}_4 \times \mathbb{Z}_4, +) \subseteq \mathrm{Aut}(\mathcal{D})$ is sharply transitive. Then Theorem 2.48 gives that there exists a $(16, 6, 2)$-difference set in $(\mathbb{Z}_4 \times \mathbb{Z}_4, +)$ from which $\mathcal{D}$ can be developed. In this particular case we have already seen such a difference set in Example 2.42.

It should be mentioned that also some BIBDs with $v < b$ can be constructed via difference sets, but in such cases we use a set of two or more *supplementary difference sets*, also called a *difference family*.

2.3.4 Existence of symmetric designs

We have already seen that there exists a projective plane, i.e., a $(q^2+q+1, q+1, 1)$-SBIBD, for each prime power q, and that there exists a Hadamard 2-design, i.e., a $(q, (q-1)/2, (q-3)/4)$-SBIBD, for each prime power $q \equiv 3 \pmod 4$, $q \geq 7$. Besides these, there are quite a few ingenious constructions of families and special examples of SBIBDs (cf, [14], p. 116). But we will focus on the necessary conditions given in the "Bruck–Ryser–Chowla Theorem", the strongest result know when considering existence of these designs.

Theorem 2.49. (*Bruck–Ryser–Chowla Theorem*). *Suppose there exists a symmetric balanced incomplete block design with parameters (v, k, λ).*

1. *If v is even, then $k - \lambda$ is a perfect square;*
2. *If v is odd, then there are integers x, y, and z, not all zero, such that*

$$x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2.$$

Theorem 2.49 was first proved for $(v, k, 1)$-SBIBDs, i.e. projective planes, by Bruck and Ryser (1949) and then generalized to (v, k, λ)-SBIBDs by Chowla and Ryser. The part of the theorem pertaining to even v was first obtained by Schützenberger [37].

By using the Bruck–Ryser–Chowla Theorem, we can prove non-existence for many SBIBD candidates with parameter sets that satisfy the fundamental identities of Proposition 2.5.

Example 2.50. *We will prove that there is no $(29, 8, 2)$-SBIBD. Theorem 2.49 gives that a necessary condition for the existence of such a design is that the following equation has a solution in integers x, y and z, not all zero.*

$$x^2 = 6y^2 + 2z^2 \tag{2.1}$$

We see that $2|x^2$ which also means that $2|x$. Let $x = 2x_1$ and we can write

$$2x_1^2 = 3y^2 + z^2, \tag{2.2}$$

which considered modulo 3 becomes $2x_1 \equiv z^2 \pmod 3$*. As 2 is not a square in* $\mathbb{Z}_3$*, this means that* $x_1 \equiv 0 \pmod 3$ *and consequently* $z \equiv 0 \pmod 3$*. Let* $x_1 = 3x_2$ *and* $z = 3z_1$*. Then Equation 2.2 can be written*

$$6x_2^2 = y^2 + 3z_1^2.$$

We see that also $3|y$*. Let* $y = 3y_1$*, then the above equation becomes*

$$2x_2^2 = 3y_1^2 + z_1^2,$$

and we are back in Equation 2.2. As this process can be repeated infinitely often we conclude that Equation 2.1 has only the trivial solution $(x, y, z) = (0, 0, 0)$*, and no* $(29, 8, 2)$*-SBIBD exists.*

At one time it was even conjectured that the necessary conditions of Theorem 2.49, together with those of Proposition 2.5, are sufficient for the existence of a SBIBD. However, this is not true which became clear when Lam et al. [24] proved that no projective plane of order 10 exists, i.e., no $(111, 11, 1)$-SBIBD exists, even though the parameters satisfy the conditions of Theorem 2.49. So far, this is the only example that shows that these conditions are not sufficient.

Among small examples of potential SBIBDs where the parameter sets satisfy Theorem 2.49, but where the existence is undecided is $(157, 13, 1)$, i.e., a projective plane of order 12. For *biplanes*, i.e., $(v, k, 2)$-SBIBDs, the only values of k for which a design is known to exist are $k = 4, 5, 6, 9, 11, 13$, and the smallest biplane for which the existence is undecided has parameters $(121, 16, 2)$. For the existence of SBIBDs with $\lambda > 2$ we have a similar situation, which is expressed in the following conjecture.

Conjecture 2.51. *(cf. [14], p. 111) For every* $\lambda > 1$*, there exists only finitely many* (v, k, λ)*-SBIBDs.*

Chapter 3

Row-column designs

3.1 Definitions and properties

Among row-column designs, we are primarily interested in a class called *triple arrays* but we will also consider some of their predecessors and related designs.

Definition 3.1. *A* row-column design $\mathcal{A}$ *is an* $r \times c$ *array in which each cell contains exactly one element of some* v*-set* V *of symbols.* $\mathcal{A}$ *is called* binary *if there is no repetition in any row or column, and is called* equireplicate *if every element of* V *appears the same number of times in* $\mathcal{A}$.

The prototypical example of a row-column design is a latin square.

Definition 3.2. *A* latin square L *of* order n *is an* $n \times n$ *array in which each one of* n *symbols occurs once in each row and once in each column.*

$$\begin{array}{cccc} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{array}$$

Figure 3.1: A latin square of order 4.

Latin squares have many applications and are studied intensively. Also, they are commonly used by a lot of people every day, as the solution of a sudoku puzzle is a latin square.

The main difference between block designs and row-column designs is that block designs have two *constraints*: points and blocks, whereas row-column designs have three: rows, columns and symbols. In a latin square, each row is incident with every column, and we say that the constraint rows are *orthogonal* with respect to the constraint columns. In fact, any pair of constraints in a latin square are orthogonal. This explains many of the nice properties of latins squares, but also that they are "expensive" as designs. There is a need for row-column designs which are non-orthogonal, but still have nice properties. One such property that involves all three constraints is called *adjusted orthogonality*.

Definition 3.3. *A binary, equireplicate row-column design with replication number* k *is said to be* adjusted orthogonal *if*

$$MN^T = kJ,$$

where M and N are the row-symbol and column-symbol incidence matrices of the design and J is the all-one matrix of appropriate order.

Hence, in a row-column design with replication number k, adjusted orthogonality means that each pair of a row and a column intersect in k symbols. In experiments, one often uses component designs formed by rows-symbols and column-symbols respectively. About relevance of adjusted ortogonaliy for this we quote John and Eccleston [20] who wrote: *"If a row-column design has the property of adjusted orthogonality then one need consider the component designs only."*

3.1.1 Youden squares

Youden's [44], who did studies on tobacco mosaic virus, realized the need for row-column designs and defined a class based on symmetric designs. Shrikhande [39] explains:

> *"Sometimes in a design the position within the block is important as a source of variation, and the experiment gains in efficiency by eliminating the positional effect. The classical example is due to Youden in his studies on tobacco mosaic virus [44] in 1937. He found that the response to treatments also depends on the position of the leaf on the plant. If the number of leaves is sufficient so that every treatment can be applied to one leaf of a tree, then we get an ordinary Latin square, in which the trees are columns and the leaves belonging to the same position constitute the rows. But if the number of treatments is larger than the number of leaf positions available, then we must have incomplete columns. Youden used a design in which the columns constituted a balanced incomplete block design, whereas the rows were complete. These designs are known as Youden squares and can be used when two-way elimination of heterogeneity is desired."*

Definition 3.4. *A* Youden square *is an arrangement of v symbols in k rows and v columns such that*

1. *every symbol occurs exactly once in each row;*
2. *the columns form a (v, k, λ)-SBIBD.*

Sometimes it is useful to write (2) of Definition 3.4 in the alternative way: *"any pair of columns intersects in a constant number of symbols"*.

Proposition 3.5. *(cf. [19], p. 30) An incidence structure having v points and v blocks, constant block size k, and constant intersection size between any two distinct blocks is a (v, k, λ)-SBIBD.*

Example 3.6. *A 4×7 Youden square in which each pair of distinct points occur together in two columns, and each pair of distinct columns intersect in two symbols.*

$$\begin{array}{ccccccc} 3 & 6 & 1 & 4 & 0 & 5 & 2 \\ 6 & 2 & 4 & 0 & 3 & 1 & 5 \\ 1 & 4 & 6 & 2 & 5 & 3 & 0 \\ 2 & 5 & 0 & 3 & 6 & 4 & 1 \end{array}$$

About existence we have the following theorem.

Theorem 3.7. *A Youden square can always be constructed from a SBIBD.*

The above theorem was first proved by Smith and Hartley [41], but we will quote a proof from Raghavarao [35]. This because it uses systems of distinct representatives (SDRs), and this approach will be of further interest later on. First we remind of the definition and a well-known result for SDRs.

Definition 3.8. *Let $A_1, A_2, \ldots, A_n$ be sets. A* system of distinct representatives *(SDR) for these sets is an n-tuple $(x_1, x_2, \ldots, x_n)$ of elements with the properties*

1) $x_i \in A_i$, for $i = 1, 2, \ldots n$;

2) $x_i \neq x_j$, for $i \neq j$.

Definition 3.9. *For any set $J \subseteq \{1, 2, \ldots, n\}$ we define*

$$A(J) = \cup_{j \in J} A_j.$$

Theorem 3.10. *(P. Hall 1935). A necessary and sufficient condition for the existence of an SDR for the collection of finite sets $A_1, A_2, \ldots, A_n$ is that*

$$|A(J)| \geq |J| \quad \text{for all } J \subseteq \{1, 2, \ldots n\}.$$

Proof of Theorem 3.7. Let $\mathcal{D}$ be a (v, k, λ)-SBIBD. Write the blocks of $\mathcal{D}$ as columns and note that the replication number r is equal to k by Proposition 2.5. Then any h columns, $1 \leq h \leq v$, contain between them hr symbols of which at least h are distinct as each symbol can occur at most r times in these h columns. Thus, by Theorem 3.10, an SDR exists for the v columns, and this SDR will be a permutation of the v symbols of $\mathcal{D}$. Bring this SDR to the first row. Deleting the first row we find that each column now contains $r - 1$ symbols, and h of these columns contain $h(r - 1)$ symbols of which at least h are distinct. Hence another SDR exists for the columns, and we bring this SDR to the second row. Continuing similarly, we can prove that the k rows can be so arranged that every symbol occurs exactly once in each row. □

Remark 3.11. *A Youden square is obviously not a square array and some authors use the name* Youden rectangle *instead. However, the reason for the term square is said to be the use of another representation with a $v \times v$ array, each of whose entries is either blank or one of k symbols, each symbol occurring exactly once in each row and in each column, every pair of rows (or columns) being occupied simultaneously in λ columns (or rows).*

3.1.2 Binary pseudo-Youden designs

The usual latin square and Youden square designs have been generalized to give other types of designs. A binary example is the following class.

Definition 3.12. *A binary row-column design where the rows and columns together form a BIBD is called a* binary pseudo-Youden design.

As a block design is formed, we know that the number of rows are equal to the number of columns. We denote such an $r \times r$ binary pseudo-Youden design on v elements by $PYD(v : r \times r)$.

Example 3.13. *A binary $PYD(9 : 6 \times 6)$. The rows and columns together form a $(9, 12, 8, 6, 5)$-BIBD.*

1	2	3	4	5	6
7	8	9	1	2	3
5	4	7	9	6	8
8	1	2	5	7	4
6	9	4	3	1	7
3	6	5	8	9	2

The class was named by Cheng [12], but the small $PYD(9:6\times 6)$ has been used and studied since the 1950s, by Kshirsagar [23], Preece [32] and others. McSorley and Phillips [28] gave a complete enumeration of it and found that there are 696 non-isomorphic $PYD(9:6\times 6)$.[1]

By combining the structures of Youden squares and affine planes, Cheng [13] was able to prove the existence of an infinite family of PYDs.

Theorem 3.14. *([13]). Let s be a prime or a prime power with $s \equiv 3 \pmod 4$. Then there exists a binary pseudo-Youden design with $v = s^2$ and $r = \frac{s(s+1)}{2}$.*

Taking $s = 3$ in Theorem 3.14 gives a $PYD(9:6\times 6)$ which is the smallest member in this family, and to the best of our knowledge, all binary $PYD(v:r\times r)$ belong to this family.

3.1.3 Triple arrays and balanced grids

In the 1960s Agrawal [2] amongst others, started to construct some row-column designs for two-way elimination of heterogenity that we now call *triple arrays*. In these designs, strong properties co-exist as two BIBDs are merged together in such a way that the design is adjusted orthogonal. We will look at the earliest example we have seen, an application example by Potthoff [30]. Note that this experiment was not made for its own sake, but purely to present the design and to illustrate the analysis part.

Example 3.15. *The experiment is to measure traffic flow at 10 different points around the campus in the mornings. The observations were made every 10 minutes between 8 a.m. and 9 a.m. on 5 mornings in september 1961. A given observation consisted of counting the number of vehicles passing the specified point during a 5-minute period. The raw results of the experiment are given in the following table. For example,* 72 *vehicles passed location* (1) *Monday between* 8.00 *and* 8.05*. A complete experiment would consist*

Days	8.00	8.10	8.20	8.30	8.40	8.50
Monday	72(1)	101(6)	59(3)	53(4)	10(8)	78(10)
Tuesday	49(2)	50(1)	98(9)	92(10)	38(5)	12(8)
Wednesday	62(3)	13(8)	49(7)	50(2)	73(9)	54(4)
Thursday	52(4)	35(7)	89(1)	82(9)	46(6)	67(5)
Friday	57(5)	55(2)	100(10)	46(6)	34(3)	48(7)

Table 3.1: Traffic flow around campus.

of 300 observations, but Potthoff used a design constructed in such a way that only 30 observations were required for the analysis. Let us look at the bare design.

1	6	3	4	8	10
2	1	9	10	5	8
3	8	7	2	9	4
4	7	1	9	6	5
5	2	10	6	3	7

Figure 3.2: The underlying design of Potthoff's experiment on traffic flow.

[1] Two binary PYDs $\mathcal{A}_1$ and $\mathcal{A}_2$ on the same set of symbols are isomorphic if $\mathcal{A}_2$ can be obtained from $\mathcal{A}_1$ by a permutation of its symbols, rows or columns.

This is a triple array. We will see that both rows and columns as sets constitutes duals of balanced incomplete block designs, merged so that every row-column pair intersects in a constant number of symbols.

Definition 3.16. *Let $\mathcal{A}$ be a binary $r \times c$ row-column design on v symbols, equireplicate with replication number k, where $k < r, c$, and let $\lambda_{rr}, \lambda_{cc}$ and λ_{rc} be positive integers. If $\mathcal{A}$ satisfies that*

1. *any two distinct rows intersect in λ_{rr} symbols;*
2. *any two distinct columns intersect in λ_{cc} symbols;*
3. *any row and column intersect in λ_{rc} symbols;*

then $\mathcal{A}$ is called a triple array*, denoted by $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$. An array as above that satisfies conditions 1–2 is called a* double array *and is denoted by $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$.*

Note that property (3) in Definition 3.16 means that $\mathcal{A}$ is adjusted orthogonal. Also, properties (1) and (2) in this context give dual conditions for BIBDs, so we have two equivalent definitions at our disposal.

Definition 3.17. *Let $\mathcal{A}$ be an $r \times c$ row-column design on v symbols that satisfies*

1. *the rows are the dual of a BIBD,*
2. *the columns are the dual of a BIBD,*
3. *every row intersects every column in a constant number of symbols,*

then $\mathcal{A}$ is called a triple array*.*

Similar to Fisher's inequality 2.9 for BIBDs, there is a useful inequality for triple arrays.

Theorem 3.18. *([4][26]) Any triple array $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ satisfies,*

$$v \geq r + c - 1.$$

The triple array we saw in Example 3.15 lies in the extremal case where $v = r+c-1$. Also, Agrawal [2] worked and suggested a construction method for triple arrays in this case, starting from SBIBDs. We will look at this construction method in Section 3.2 but first we will consider the opposite direction, and for this we use the following observation.

Lemma 3.19. *([26]) Suppose $\mathcal{A}$ is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with $v = r + c - 1$. Then*

$$\lambda_{cc} = r - \lambda_{rc} = v - 2c + \lambda_{rr} + 1.$$

Theorem 3.20. *([26]) Suppose $\mathcal{A}$ is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with $v = r + c - 1$. Then there exists a $(v + 1, r, \lambda_{cc})$-SBIBD.*

Proof. Label the rows of $\mathcal{A}$ by $i = 1, 2, \ldots, r$, the columns by $j = r+1, r+2, \ldots, r+c$, and let R_i and C_j denote the support of row i and column j respectively. We construct a $(v + 1, r, \lambda_{cc})$-SBIBD $\mathcal{D} = (X, \mathcal{B})$ by taking these labels as point set, so $X = \{1, 2, \ldots, r + c\}$, and we construct one block for each symbol s of $\mathcal{A}$ by:

$$B_s = \{i : s \notin R_i\} \cup \{j : s \in C_j\} \quad \text{for } s = 1, 2, \ldots, v,$$

and also add the block

$$B_0 = \{1, 2, \ldots, r\}.$$

We note that $|X| = |\mathcal{B}| = v + 1$ and that the block size of $\mathcal{D}$ is constant as any given s does not occur in $r - k$ rows but does occur in k columns of $\mathcal{A}$, so $|B_s| = (r - k) + k = r = |B_0|$. $\mathcal{D}$ is equireplicate because any point i will be in a B_s when s does not occur in R_i. This happens in $v - c$ cases and in the block B_0, so i will be in $v - c + 1 = r$ blocks. Also, a point j occurs in different B_s for each of the r symbols in C_j. It remains to show that $\mathcal{D}$ is balanced and we have three cases of pairs of points.

$\{i_1, i_2\}$: Given any two distinct rows i_1 and i_2 of $\mathcal{A}$, the sieve principle gives that $|(R_{i_1} \cup R_{i_2})^{\complement}| = v - 2c + \lambda_{rr}$. As the pair $\{i_1, i_2\}$ also occurs in B_0, it will be in a total of $v - 2c + \lambda_{rr} + 1$ blocks of $\mathcal{D}$, and Lemma 3.19 gives that $v - 2c + \lambda_{rr} + 1 = \lambda_{cc}$.

$\{j_1, j_2\}$: A pair $\{j_1, j_2\}$ will meet in λ_{cc} blocks of $\mathcal{D}$ as $|C_{j_1} \cap C_{j_2}| = \lambda_{cc}$.

$\{i, j\}$: A pair $\{i, j\}$ will meet in $|R_i^{\complement} \cap C_j| = r - \lambda_{rc}$ blocks of $\mathcal{D}$, and Lemma 3.19 gives that $r - \lambda_{rc} = \lambda_{cc}$. □

The component designs of the triple array correspond to subdesigns of the constructed SBIBD which was pointed out by [27].

Corollary 3.21. *Let $\mathcal{A}$ be a triple array with $v = r + c - 1$ and let $\mathcal{D}$ be the $SBIBD$ constructed from $\mathcal{A}$ as in Theorem 3.20. Then*

1. *the dual design of the rows in $\mathcal{A}$ is the complementary design of the derived design of $\mathcal{D}$ with respect to B_0,*
2. *the dual design of the columns in $\mathcal{A}$ is the residual design of $\mathcal{D}$ with respect to B_0.*

The converse of Theorem 3.20 is open, and is known as *Agrawal's Conjecture*.

Conjecture 3.22. *([2]). If there exists a $(v+1, r, \lambda_{cc})$-SBIBD with $r - \lambda_{cc} > 2$ then there exists a $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$ with $v = r + c - 1$.*

When we examine construction methods in Section 3.2, we will take a closer look at Agrawal's Conjecture and we will also see that there are many triple arrays with $v = r + c - 1$. However, when $v > r + c - 1$, there is only one know triple array. It is a $TA(35, 3, 5, 1, 3 : 7 \times 15)$ that was asked for by Preece [31] already in 1976, but was found much later by McSorley et al. [26] and whose structure was studied by Yucas [45].

We now turn our attention to a class of designs which is quite unknown, but in fact contains almost every row-column design mentioned in this thesis. They are called *balanced grids* and were introduced by McSorley et al. [26] in 2005. In balanced grids we consider how pairs of distinct symbols occur together. Let x and y be two symbols in a binary row-column design. Let r_{xy} denote the number of rows where both x and y occur and c_{xy} the number of columns where both x and y occur.

Definition 3.23. *Let $\mathcal{A}$ be a binary row-column design and define $\mu_{xy} = r_{xy} + c_{xy}$. If there is a constant μ such that $\mu_{xy} = \mu$ for every distinct x and y, then $\mathcal{A}$ will be called a* balanced grid.

Theorem 3.24. *([26]). An $r \times c$ balanced grid based on v symbols satisfies*

$$\mu = \frac{rc(r + c - 2)}{v(v - 1)},$$

moreover, it will be equireplicate, with replication number

$$k = \frac{rc}{v}.$$

A balanced grid is denoted by $BG(v, k, \mu : r \times c)$.

There is an inequality also for balanced grids.

Theorem 3.25. *([26]) Any balanced grid $BG(v, k, \mu : r \times c)$ satisfies $v \leq r + c - 1$.*

That the inequalities for triple arrays and balance grids are both extremal when $v = r + c - 1$ suggest the following result.

Theorem 3.26. *([27]) Let $v = r + c - 1$. Then every triple array is a $TA(v, k, c - k, r - k, k : r \times c)$ and every balanced grid is a $BG(v, k, k : r \times c)$, and they are equivalent.*

This theorem is useful as it gives us an alternative view and definition of triple arrays in the canonical case. But let us remember that besides these triple arrays, also latin squares, Youden squares and, as was pointed out in [28], pseudo-Youden designs all are balanced grids.

3.2 Two construction methods for triple arrays

Quite a few triple arrays are known. A comprehensive list of these and many examples can be found in [26], and there is also a database [46] from which triple arrays can be downloaded. However, the existence question is still open. What we have is a general construction for the canonical case $v = r + c - 1$ called *Agrawal's method*. This method would give all such triple arrays, but one of the steps of the construction is unproved and is left to trial and error solutions. Besides Agrawal's method, we have the family of *Paley triple arrays*, which is the single known infinite family of triple arrays. In the non-extremal case when $v > r + c - 1$, the only triple array known is the $TA(35, 3, 5, 1, 3 : 7 \times 15)$ in [26] as we have mentioned before.

3.2.1 Agrawal's method

In 1966, Agrawal [2] gave a construction method for triple arrays starting from $(v+1, r, \lambda_{cc})$-SBIBDs. He could not prove the method, although he found no counterexample provided that $r - \lambda_{cc} > 2$.

Given a (v, k, λ)-SBIBD $\mathcal{D}$ and a block B of $\mathcal{D}$ we know by Proposition 2.21 that the residual design $\mathcal{D}^B$ is a $(v-k, v-1, k, k-\lambda, \lambda)$-BIBD, and the following lemma gives us a simular result for the complementary design of the derived design $\mathcal{D}^B$, provided that $\mathcal{D}$ is non-trivial.

Lemma 3.27. *Let $\mathcal{D} = (X, \mathcal{B})$ be a (v, k, λ)-SBIBD with $v - 1 > k$. Then $(\mathcal{D}_B)'$, the complementary design of the derived design of $\mathcal{D}$ with respect to some block $B \in \mathcal{B}$, is a $(k, v - 1, v - k, k - \lambda, v - 2k + \lambda)$-BIBD.*

Proof. A moment of thought gives the four first parameters of $(\mathcal{D}_B)'$, and the sieve principle gives that its index must be $v - 2k + \lambda$. So what we need to prove is that $v - 2k + \lambda > 0$. Let $x, y \in X$. As $2k - \lambda$ is the number of blocks that contain one or both of x and y, we cannot have $v - 2k + \lambda < 0$. So let us assume that $v - 2k + \lambda = 0$, and we will show that this can only happen if $v = k + 1$. We use Proposition 2.5 to develop

$$v - 2k + \lambda = \frac{\lambda(v-1)(v-2k)}{k(k-1)} + \frac{\lambda k(k-1)}{k(k-1)} = \frac{\lambda(v^2 - 2vk - v + 2k + k^2 - k)}{k(k-1)} = \\ = \frac{\lambda\left((v-k)^2 - (v-k)\right)}{k(k-1)} = \frac{\lambda(v-k)(v-k-1)}{k(k-1)},$$

which is equal to 0 if and only if $v = k + 1$. □

These two BIBDs seem to have promising parameters for being merged together as they have the same number of blocks, same block size and mirror each other when it comes to the number of points and replication number. Further, Agrawal [1] had made the following observation.

Lemma 3.28. *([1]) Let $\mathcal{D}$ be a (v, k, λ)-SBIBD. Let N_1 denote the incidence matrix of the residual design of $\mathcal{D}$ with respect to a block B, and let N_2 denote the incidence matrix of the complementary design of the derived design of $\mathcal{D}$ with respect to the same block B. Then*

$$N_1 N_2^T = (k - \lambda) J_{v-k,k},$$

where J is an $(v - k) \times k$ all-one matrix.

A comparison of this result with Definition 3.3 suggests that it might be possible to merge this pair of BIBDs into a row-column design with adjusted orthogonality.

Construction 3.29. *Agrawal's method [2].*

1. *Take a $(v + 1, r, \lambda_{cc})$-SBIBD with $r - \lambda_{cc} > 2$ and label the blocks $B_0, B_1, \ldots, B_v$.*
2. *Denote the blocks of the residual design $\mathcal{D}^{B_0}$ by $B'_s = B_s \setminus B_0$, $s = 1, 2, \ldots, v$, and label its elements $j = 1, 2, \ldots, c$, where $c = v + 1 - r$.*
3. *Let N_2 denote the incidence matrix of the complement of the derived design $\mathcal{D}_{B_0}$ with blocks $B''_s = B_0 \setminus B_s$, $s = 1, 2, \ldots, v$, and elements labeled $i = 1, 2, \ldots, r$.*
4. *For each column $s = 1, 2, \ldots, v$, of N_2, replace the entries 1 by the elements of the block B'_s in any order, and let the remaining cells be undefined.*
5. *Rearrange the elements within each column s, using only the defined cells, so that each element of $\mathcal{D}^{B_0}$ occurs exactly once in every row. Then we have an $r \times v$ array A where $A(i, s) = j$ in rc defined cells, (A is called the RL-form of the triple array).*
6. *Map the defined triplets (i, s, j) from A to the $r \times c$ array C where $C(i, j) = s$. Then C is a $r \times c$ triple array.*

Note that it is not proven that the rearranging step (5) of Agrawal's method can always be done. Ragesvarao and Nageswararao [36] claimed that they had proven it, using an argument with systems of distinct representatives, similary to the proof for Youden squares in Theorem 3.7. But this proof is flawed, as was pointed out by Wallis and Yucas [43].

We will write out a proof of Agrawal's method where we assume that the rearrangment step (5) can be carried out. But first we use Lemma 3.27 and Proposition 2.21 to simplify the parameter expressions of $\mathcal{D}^B$ and the complementary design of $\mathcal{D}_B$ of a $(v + 1, r, \lambda_{cc})$-SBIBD.

Lemma 3.30. *Let $\mathcal{D}$ be a $(v+1, r, \lambda_{cc})$-SBIBD and let B be a block of $\mathcal{D}$. If we write $c = v+1-r$, $k = r - \lambda_{cc}$ and $\lambda_{rr} = v - 2r + \lambda_{cc} + 1$, then the complementary design of $\mathcal{D}_B$ is a $(r, v, c, k, \lambda_{rr})$-BIBD, and $\mathcal{D}^B$ is a $(c, v, r, k, \lambda_{cc})$-BIBD.*

Proposition 3.31. *(For Agrawal's method) Let $\mathcal{D}$ be a $(v+1, r, \lambda_{cc})$-SBIBD with $r-\lambda_{cc} > 2$ and assume that the rearranging step (5) of Construction 3.29 can always be carried out. Then the array C of Construction 3.29 is a triple array.*

Proof. Suppose it is possible to rearrange the elements as specified in step (5) of Construction 3.29. The $r \times v$ array A has rc defined cells which means that C is an $r \times c$ row-column design with v symbols. C is equireplicate with replication number $r - \lambda_{cc}$ as each block B''_s has $r - \lambda_{cc}$ elements. That C is binary follows from the rearranging step (5).

N_2 is the incidence matrix for points and blocks of the complementary design of $\mathcal{D}_{B_0}$. When we replace the entries 1 in step (4) and rearrange in (5), it does not affect what rows and columns are incident. So after the transcription to C, when columns and symbols are interchanced, N_2 will serve as incidence matrix for rows–symbols in C. Similary, N_1 is the incidence matrix for points and blocks of $\mathcal{D}^{B_0}$. Moving these points (symbols) within columns to N_2 does not affect what points (symbols) and columns are incident. Thus, N_1 is the incident matrix for symbols–columns in A and after the transcription to C, N_1 will serve as the incident matrix for columns–symbols in C.

By Lemma 3.30, N_2 is the incidence matrix of a $(r, v, c, k, \lambda_{rr})$-BIBD and N_1 is the incidence matrix of a $(c, v, r, k, \lambda_{cc})$-BIBD, so both rows and columns form duals of BIBDs respectively. Hence, C is a double array.

Let $M_{(rs)}$ and $M_{(cs)}$ be incidence matrices for rows-symbols and columns-symbols of C respectively, then by Lemma 3.28

$$M_{(cs)} M_{(rc)}^T = N_1 N_2^T = (r - \lambda_{cc}) J_{c,r}.$$

Hence, C is adjusted orthogonal so C is a triple array. □

Example 3.32. *Let us construct a 5×6 triple array by Agrawal's method. We start with a $(11, 5, 2)$-SBIBD $\mathcal{D}$, here given by its incidence matrix N. We have labeled the blocks/columns, and to make it more easy to see the structure we have written N so that B_0 consists of the first five elements, emphasized the 1's in that column and used the sign $(-)$ instead of the entry (0).*

$$N = \begin{array}{ccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline
\mathbf{1} & - & - & 1 & - & - & - & 1 & 1 & 1 & - \\
\mathbf{1} & - & 1 & - & - & 1 & - & - & - & 1 & 1 \\
\mathbf{1} & 1 & - & 1 & - & - & 1 & - & - & - & 1 \\
\mathbf{1} & 1 & 1 & - & 1 & - & - & 1 & - & - & - \\
\mathbf{1} & - & - & - & 1 & 1 & 1 & - & 1 & - & - \\
- & 1 & - & - & 1 & - & - & - & 1 & 1 & 1 \\
- & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - \\
- & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - \\
- & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 \\
- & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - \\
- & - & 1 & - & - & - & 1 & 1 & 1 & - & 1
\end{array}$$

Let N_1 denote the incidence matrix for the residual design $\mathcal{D}^{B_0}$. It is a $(6, 3, 2)$-BIBD and we label the elements $j = 1, 2, \ldots, 6$. Let N_2 denote the incidence matrix for the *complement of the derived design $\mathcal{D}_{B_0}$. It is a $(5,3,3)$ BIBD and we label the elements $i = 1, 2, \ldots, 5$.*

Let the elements j from block B'_s of $\mathcal{D}^{B_0}$ replace the 1's in column s of N_2 for every $s = 1, 2, \ldots, v$.

	1	2	3	4	5	6	7	8	9	10
1	1	6		1	2	6				6
2	5		2	3		3	5	2		
3		3		4	5		6	6	5	
4			4		4	5		1	3	4
5	2	2	3				4		1	1

Rearrange the elements within columns so that every element occurs exactly once in every row. Then we have the array A where $A(i,s) = j$, (the RL-form of the triple array).

$A =$

	1	2	3	4	5	6	7	8	9	10
1	5	6		1	2	3				4
2	1		3	4		6	5	2		
3		2		3	5		4	6	1	
4			2		4	5		1	3	6
5	2	3	4				6		5	1

Construct the array C by $C(i,j) = s$. It is a $TA(10,3,3,2,3 : 5 \times 6)$.

$C =$

	1	2	3	4	5	6
1	4	5	6	10	1	2
2	1	8	3	4	7	6
3	9	2	4	7	5	8
4	8	3	9	5	6	10
5	10	1	2	3	9	7

Agrawal [2] wrote that his method did not seem to work when starting with a (v,k,λ)-SBIBD in which $k - \lambda \leq 2$. However, the condition $k - \lambda > 2$ does not exclude any triple array that can be constructed in any other way.

Observation 3.33. *A (v,k,λ)-SBIBD with $k - \lambda = 2$ is either a $(7,3,1)$-SBIBD or its complementary design, a $(7,4,2)$-SBIBD.*

Proof. As $k - \lambda = 1$ implies the trivial case $k = v - 1$ we assume that $k - \lambda = 2$. By Proposition 2.5 we can write $(k-2)(v-1) = k(k-1)$. So, $k - 2$ divides $k(k-1)$ and long division gives

$$v - 1 = k + 1 + \frac{2}{k-2}.$$

Hence, k can only take values in $\{3, 4\}$ which in both cases give $v = 7$. □

These SBIBDs would correspond to a $TA(6,2,2,1,2 : 3 \times 4)$ which is known to not exist by exhaustion.

3.2.2 Paley triple arrays

"God loves odd numbers."
Vergilius, Eclogae

In Example 3.15, we could see that a 5×6 triple array was used for the experiment on traffic flow. Besides this design, Potthoff [30] constructed a few more triple arrays, all of size $q \times (q+1)$ where q is an odd prime. Under the years, several constructions have been published on the $q \times (q+1)$ theme. We can mention Agrawal [3], Preece [33], Seberry [38] and Bagshi [4]. In 2005, Preece et al. [34] finally gave a construction for $q \times (q+1)$ triple arrays, which holds for every odd prime power $q \geq 5$.

Let q be an odd prime power, and let Q denote the set of non-zero squares in $GF(q)$ and N denote the set of non-squares in $GF(q)$. Further, let $Q_0 = Q \cup \{0\}$ and $N_0 = N \cup \{0\}$.

Construction 3.34. *([34]) Order the elements of $GF(q)$, say by $\{0 = w_0, w_1, w_2, \ldots, w_{q-1}\}$, and let $GF(q)' = \{0' = w_0', w_1', w_2', \ldots, w_{q-1}'\}$ be a duplicate copy. For non-zero elements a and b in $GF(q)$, define the $q \times q$ matrix C_0 by:*

$$C_0(i,j) = \begin{cases} w_i - \frac{w_i - w_j}{a} & \text{if } w_i - w_j \in Q \\ (w_i + \frac{w_i - w_j}{b})' & \text{if } w_i - w_j \in R_0 \end{cases}.$$

Let C be the $q \times (q+1)$ matrix obtained by appending $(w_0, w_1, \ldots, w_{q-1})$ to C_0 as column q, i.e. $C(i,q) = w_i$ for $i = 0, 1, \ldots, q-1$.

If C in Construction 3.34 is a triple array, then it is said to be a *Paley triple array*. But for this to happen we need to choose a and b in an appropriate way.

Theorem 3.35. *[34] Suppose $q \equiv 1 \pmod 4$. Choose a and b such that $ab \in Q$, $(a-1) \in Q$ and $(b+1) \in N$. Then C is a Paley triple array.*

Theorem 3.36. *[34] Suppose $q \equiv 3 \pmod 4$. Choose a and b such that $(a-1)(b+1) \in Q$ and if $(a-1) \in N$ then $ab \in Q$. The C is a Paley triple array.*

Preece et al. [34] proved that for any odd prime power $q \equiv 5$, then a and b can always be chosen in the required way. Further, they diveded the infinite family of Paley triple arrays into eight different types depending on q and the choises for a and b.

Type	Description
$T_{1,1}$:	$q \equiv 1 \pmod 4$, $a \in Q$,
$T_{1,2}$:	$q \equiv 1 \pmod 4$, $a \in N$,
$T_{3,1}$:	$q \equiv 3 \pmod 4$, $a, b \in Q$ and $(a-1) \in Q$,
$T_{3,2}$:	$q \equiv 3 \pmod 4$, $a, b \in Q$ and $(a-1) \in N$,
$T_{3,3}$:	$q \equiv 3 \pmod 4$, $a, b \in N$ and $(a-1) \in Q$,
$T_{3,4}$:	$q \equiv 3 \pmod 4$, $a, b \in N$ and $(a-1) \in N$,
$T_{3,5}$:	$q \equiv 3 \pmod 4$, $a \in Q$ and $b \in N$,
$T_{3,6}$:	$q \equiv 3 \pmod 4$, $a \in N$ and $b \in N$.

Table 3.2: The eight types of Paley triple arrays. Note that the conditions here are minimal for identifying a type and imply further conditions by Theorems 3.35 and 3.36.

Example 3.37. *We will construct a $TA(18, 5, 5, 4, 5 : 9 \times 10)$*[2]*. Let γ be a root of the primitive polynomial $x^2 + x + 2$ over $GF(3)$. Then $GF(9) = \{0, 1, 2, \gamma, \gamma+1, \gamma+2, 2\gamma, 2\gamma+1, 2\gamma+2\}$. That the irreducible polynomial x^2+x+2 is primitive over $GF(3)$ means that γ generates the multiplicative group of $GF(9)$, as we can see in the following list.*

$$\begin{array}{llll} \gamma^1 = \gamma, & \gamma^2 = 2\gamma+1, & \gamma^3 = 2\gamma+2, & \gamma^4 = 2, \\ \gamma^5 = 2\gamma, & \gamma^6 = \gamma+2, & \gamma^7 = \gamma+1, & \gamma^8 = 1. \end{array}$$

[2] This example, on which we will continue in Section 4, was also given by [34].

Take $a = \gamma^4$ *and* $b = \gamma^6$. *Then* $ab = \gamma^2 \in Q$, $(a-1) = \gamma^8 \in Q$ *and* $(b+1) = \gamma \in N_0$. *Theorem 3.35 gives that these choices will give a Paley triple array of type* $T_{1,1}$ *in Construction 3.34.*

First we order $GF(9)$ *by* $\{0, \gamma, \gamma^2, \ldots, \gamma^8\}$ *and note that* $a^{-1} = \gamma^4$ *and* $b^{-1} = \gamma^2$. *Then, we compute the entries in* C_0*, let us do* $C_0(5,1)$ *as an example. As* $\gamma^5 - \gamma = \gamma \in N_0$ *we have that*

$$C_0(5,1) = \left(\gamma^5 + (\gamma^5 - \gamma)\gamma^2\right)' = (\gamma^5 + \gamma^3)' = (\gamma + 2)' = \gamma^{6'}.$$

When all entries in C_0 *are in place, we append* $(0, \gamma, \gamma^2, \ldots, \gamma^8)$ *as column 9 to get the Paley triple array* C.

$$C = \begin{bmatrix} 0' & \gamma^{7'} & \gamma^6 & \gamma' & \gamma^8 & \gamma^{3'} & \gamma^2 & \gamma^{5'} & \gamma^4 & 0 \\ \gamma^{4'} & \gamma' & 0' & \gamma^8 & \gamma^{5'} & \gamma^{2'} & \gamma^7 & \gamma^6 & \gamma^3 & \gamma \\ \gamma^6 & \gamma^{8'} & \gamma^{2'} & \gamma^5 & \gamma^{7'} & \gamma^3 & 0 & \gamma' & \gamma^{4'} & \gamma^2 \\ \gamma^{6'} & \gamma^8 & \gamma^5 & \gamma^{3'} & 0' & \gamma^2 & \gamma^{7'} & \gamma^{4'} & \gamma & \gamma^3 \\ \gamma^8 & \gamma^{3'} & \gamma^{6'} & \gamma^{2'} & \gamma^{4'} & \gamma^7 & \gamma' & \gamma^5 & 0 & \gamma^4 \\ \gamma^{8'} & \gamma^{6'} & \gamma^3 & \gamma^2 & \gamma^7 & \gamma^{5'} & 0' & \gamma^4 & \gamma' & \gamma^5 \\ \gamma^2 & \gamma^7 & 0 & \gamma^{5'} & \gamma^{8'} & \gamma^{4'} & \gamma^{6'} & \gamma & \gamma^{3'} & \gamma^6 \\ \gamma^{2'} & \gamma^6 & \gamma^{3'} & \gamma^{8'} & \gamma^5 & \gamma^4 & \gamma & \gamma^{7'} & 0' & \gamma^7 \\ \gamma^4 & \gamma^3 & \gamma^{5'} & \gamma & 0 & \gamma^{7'} & \gamma^{2'} & \gamma^{6'} & \gamma^{8'} & \gamma^8 \end{bmatrix}$$

To construct C in Example 3.37 requires a lot of computations. However, when q is a prime we just have to compute one column (or row) of C_0, and then it is very easy to complete the array.

Proposition 3.38. *([34]) If* q *is a prime and* $GF(q)$ *is ordered by* $\{0, 1, 2, \ldots, q-1\}$ *then* C_0 *has cyclic transversals. That is*

$$C_0(i+k, j+k) = C(i,j) + k,$$

where row and column numbers are interpreted as integers modulo q*, when additions are involved.*

Chapter 4

Research questions and summary of papers I-III

In this chapter, we discuss the issues that formed the basis for our research and provide additional background on these. Furthermore, we describe how we approached these problems and what results we achieved.

4.1 Inner balance?

Which triple arrays are balanced for intersection and which symmetric incomplete block designs have inner balance?
That a triple array is adjusted orthogonal means that its row-column intersections form an incomplete block design. McSorley et al. [26] found one triple array in which this block design is a BIBD. They called it a triple array *balanced for intersection*, and asked if there are any more having this property. This was our original research question for Paper I and we attacked the problem by moving it from the triple array to the corresponding SBIBD, in which we define a corresponding *inner design* and say that the SBIBD has *inner balance* if its inner design is a BIBD. A motivation for this research, besides contributing to the knowledge of triple arrays and finding BIBDs, is that these inner designs are closely related to Agrawal's conjecture, and knowledge about the structure of these inner designs is probably one of the keys to this conjecture.

Example 4.1. *McSorley et al. [26] found a triple array which is balanced for intersection, a* $TA(10,3,3,2,3:5\times 6)$ *which corresponds to a* $(11,5,2)$*-SBIBD.*

1	2	4	10	5	6
4	1	6	3	7	8
8	5	2	4	9	7
9	8	3	5	6	10
10	3	9	7	1	2

The row-column intersections of this triple array can be seen in the array S *here below, and* *they form a* $(10, 30, 9, 3, 2)$*-BIBD.*

$\{1,4,10\}$	$\{1,2,5\}$	$\{2,4,6\}$	$\{4,5,10\}$	$\{1,5,6\}$	$\{2,6,10\}$
$\{1,4,8\}$	$\{1,3,8\}$	$\{3,4,6\}$	$\{3,4,7\}$	$\{1,6,7\}$	$\{6,7,8\}$
$\{4,8,9\}$	$\{2,5,8\}$	$\{2,4,9\}$	$\{4,5,7\}$	$\{5,7,9\}$	$\{2,7,8\}$
$\{8,9,10\}$	$\{3,5,8\}$	$\{3,6,9\}$	$\{3,5,10\}$	$\{5,6,9\}$	$\{6,8,10\}$
$\{1,9,10\}$	$\{1,2,3\}$	$\{2,3,9\}$	$\{3,7,10\}$	$\{1,7,9\}$	$\{2,7,10\}$

4.1.1 Summary of paper I

We move the problem of triple array balanced for intersection to the related SBIBD $\mathcal{D}$ by defining an inner design on $\mathcal{D}$.

Proposition 4.2 (Proposition 2.1 of Paper I)**.** *Let* $\mathcal{D} = (\mathcal{X}, \mathcal{B})$ *be a* (v, k, λ)*-SBIBD and let* $B_0 \in \mathcal{B}$ *be a fixed block. For* s *such that* $B_s \in \mathcal{B} \setminus \{B_0\}$*,* i *such that* $x_i \in B_0$*, and* j *such that* $x_j \in X \setminus B_0$*, let*

$$R_i = \{s : x_i \in B_0 \setminus B_s\}; \quad C_j = \{s : x_j \in B_s \setminus B_0\}.$$

Then the sets $R_i \cap C_j$ *form the blocks of an incomplete block design with parameters*

$$\left(v - 1, k(v - k), (k - \lambda)^2, k - \lambda\right),$$

which we call the inner design of $\mathcal{D}$ with respect to B_0 *and denote by* $\mathcal{D}_\star$*. If* $\mathcal{D}_\star$ *is balanced we say that* $\mathcal{D}$ *has* inner balance.

The blocks of the inner design of a SBIBD correspond to the row-column intersections of the related triple array, if it exists, and that is why knowledge of the structure of the inner design can be a key to Agrawal's conjecture.

Example 4.3. *The* $(11, 5, 2)$*-SBIBD that corresponds to the triple array in Example 4.1 here above can be found in Example 2.22 in Section 2.3. Applying Proposition 4.2 on that SBIBD with respect to the block* B_0 *gives:*

$$R_0 \cap C_5 = \{1, 2, 4, 5, 6, 10\} \cap \{1, 4, 8, 9, 10\} = \{1, 4, 10\}; \quad R_0 \cap C_6 = \{1, 2, 5\},$$

and so on, which coinside with the row-column intersections displayed in the array S *in Example 4.1 as* $S(i, j) = R_i \cap C_j$*.*

Let us for a moment discuss the relation between Agrawal's method and inner designs. Given an array S where $S(i, j) = R_i \cap C_j$ as in Example 4.1, we get a triple array A if we manage to choose one symbol in each cell such that there is no repetition in any row or column, and this is equivalent to Agrawal's method. Such an array A is called an *array of distinct representatives* (ADR). This kind of arrays was defined and studied by Fon-der-Flaass [16] in a more general setting. Given an array (S_{ij}), $1 \le i \le n$, $1 \le j \le m$ of finite sets, does there exist an array A of elements (x_{ij}) such that

- $x_{ij} \in S_{ij}$,
- $x_{ij} \neq x_{ik}$ when $j \neq k$,
- $x_{ij} \neq x_{kj}$ when $i \neq k$?

Fon-der-Flaass [16] also gave the ansver for the general ADR case.

Theorem 4.4. *([16]) The problem ADR is NP-complete. It remains NP-complete even when $n = 2$, $|S_{ij}| \leq 3$, no element appears in more than four of the sets S_{ij}, and there exist sets M_i, $1 \leq i \leq n$ and N_j, $1 \leq j \leq m$ such that $S_{ij} = M_i \cap N_j$ for all i, j.*

However, when dealing whith inner designs of SBIBDs we are far from the general case. as we have a structure with strong properties. So the problem of choosing a triple array may very well be solvable.

A general reason for moving the question from triple arrays to (v, k, λ)-SBIBDs is that the latter are more well-studied, which allow us to use strong results like the Bruck–Ryser–Chowla Theorem. But the main key that makes it possible to get a grip on the inner design is to studying the block intersections of the derived and the residual design with respect to the block B_0. We are able to show that the residual design with respect B_0 must be quasi-symmetric and give necessary and sufficient conditions on the block intersection numbers.

Theorem 4.5 (Theorem 2.8 in Paper I)**.** *Let $\mathcal{D}$ be a (v, k, λ)-SBIBD with $k - \lambda > 1$. Then $\mathcal{D}$ has inner balance with respect to a block B_0 if and only if the residual design $\mathcal{D}^{B_0}$ is quasi-symmetric with intersection numbers that are roots of the equation*

$$x^2 - (k - \lambda)x + \frac{\lambda(k - \lambda)^2(k - \lambda - 1)}{k^2 - k - \lambda} = 0.$$

The theorem above is a key result, as both quasi-symmetry and the quadratic equation give strong conditions. From this we can derive more necessary conditions and we can exclude projective planes and all biplanes, but one.

Corollary 4.6. *Let $\mathcal{D}$ be a (v, k, λ)-SBIBD with inner balance and $\lambda \leq 2$, then $\mathcal{D}$ is the unique $(11, 5, 2)$-SBIBD.*

We use our results and related results like Proposition 2.27 for quasi-symmetry to define a set Ω consisting of all parameter sets (v, k, λ) satisfying our necessary conditions for inner balance. By these results we can exclude many SBIBDs, but there is one infinite class of potential SBIBDs in Ω given by the solutions of a certain Pell-equation.

Proposition 4.7. *Given any solution (s, t) of the Pell equation $s^2 - 3t^2 = 1$ for positive integers s and t with $s > 2$ and t odd. Take $v = 4t^2 + 2, k = 2t^2 - s + 1$,
$\lambda = t^2 - s + 1$, then the triple (v, k, λ) is in the candidate set Ω.*

In a computer search for potential SBIBDs in Ω for all $\lambda \leq 10^6$, we found that besides the $(11, 5, 2)$-SBIBD there are only two potential SBIBDs that can have inner balance, and both of them belong to the Pell-class described in Proposition 4.7.

As the inner design of a SBIBD is defined with respect to a particular block we ask when a SBIBD can have inner balance with respect to every block. This leads us to consider the quasi-3 property from Definition 2.29.

Theorem 4.8 (Theorem 3.3 of Paper I)**.** *Let $\mathcal{D} = (X, \mathcal{B})$ be an SBIBD with inner balance with respect to a block B_0. Then $\mathcal{D}$ has inner balance with respect to every block $B_s \in \mathcal{B}$ if and only if $\mathcal{D}$ is quasi-3.*

The existence question for quasi-3 designs is not settled, but there is a conjecture that any quasi-3 design lies in one of the five classes in Table 4.1. We prove that there is only one SBIBD in these five classes that can have inner balance, and that is the $(11, 5, 2)$-SBIBD.

The criteria we found for SBIBDs with inner balance are also formulated for triple arrays balanced for intersection.

Class	Description
(C1)	(v, k, λ)-SBIBDs with $\lambda \leq 2$.
(C2)	$PG_{n-1}(n, q)$, (point-hyperplanes in the projective geometry $PG(n, q)$).
(C3)	SDP-designs, (designs with the symmetric difference property).
(C4)	$(4u^2, 2u^2 - u, u^2 - u)$-designs.
(C5)	The complementary designs of quasi-3 designs.

Table 4.1: The five know classes of quasi-3 designs.

Notes 4.9. There is a misprint in Paper I. In Conjecture 1.4 we have written $r - \lambda_{cc} \geq 2$, but it should be $r - \lambda_{cc} > 2$.

Author's contributions to this paper: I came up with the idea for this paper while I was working on my Master's Thesis and discussed it with P. H., my supervisor at that time. Then, I did independent research and later we wrote this paper together.

4.2 Any other families?

Are there any other row-column designs that are balanced for intersection? As triple arrays balanced for intersections seem to be quite rare, it is natural to ask if there are any other families of row-column designs with this property. This question is addressed in Paper II.

Definition 4.10. *Let $\mathcal{A}$ be a binary row-column design on v symbols. The row-column intersections of $\mathcal{A}$ as sets form a design $\mathcal{D}$, which we call the* inner design of $\mathcal{A}$*. If $\mathcal{D}$ is a BIBD, then we say that $\mathcal{A}$ is* balanced for intersection*.*

McSorley and Phillips [28] made a thorough investigation of $PYD(9 : 6 \times 6)$ where they also pointed out that every binary $PYD(v : r, r)$ is a balanced grid. This led us to check their examples and we found them to be balanced for intersection, with $(9, 36, 16, 4, 6)$-BIBDs as inner designs. We asked if all binary PYDs have this property and moved on to check the only class we know to exist, the infinite class constructed by Cheng [13] that was given in Theorem 3.14 in Section 3.1. We say that these PYDs are of *Cheng type*.

4.2.1 Summary of paper II

We proved the following key result for balanced grids.

Proposition 4.11 (Proposition 4.4 in Paper II)**.** *Let $\mathcal{A}$ be a $BG(v, k, \mu : r \times c)$ with $r \leq c < v$, which is adjusted orthogonal and where r_{xy} and c_{xy} can take at most two values, both non-zero. Then $\mathcal{A}$ is balanced for intersection.*

Cheng [13] uses Youden squares and affine planes together in an ingenious way to construct the Cheng type family. We reused this construction to prove that the sufficient conditions of Proposition 4.11 are satisfied and this gave us the main result of Paper II.

Theorem 4.12 (Theorem 4.5 in Paper II)**.** *Every pseudo-Youden design of Cheng type is balanced for intersection.*

Cheng's proof for the PYD family is an existence proof as the last rearranging step which gives the array is not constructive. However, up to that step the proof is constructive and we can construct the inner designs which form the following class of BIBDs.

Observation 4.13 (Observation 4.6 in Paper II)**.** *Let $s \equiv 3 \pmod 4$ be a prime or a prime power, then we can construct a* $\left(s^2, \frac{s^2(s+1)^2}{4}, \frac{(s+1)^4}{16}, \frac{(s+3)(s+1)^3}{64}\right)$*-BIBD.*

Author's contributions to this paper: All ideas and results in this paper were obtained by me.

4.3 A third neglected, dismissed and ignored construction

What can be done with the "Raghavarao–Nageswararao" construction method? Besides Agrawal's method and the construction of Paley triple arrays there is a third possible construction of triple arrays. It was suggested, but not investigated by Raghavarao and Nageswararao [36]. Later this method was proven to be defective through a counterexample by Wallis and Yucas [43]. However, as this method seems to work in some cases, we decided to investigate what can and what cannot be constructed by this method.

An $r \times c$ triple array A on v symbols consists of triplets (i, j, s) given by $A(i, j) = s$. If we exchange the roles of columns and symbols, we get a binary $r \times v$ array R, $R(i, s) = j$ with c defined cells in each row. This is called the RL-form of the triple array, and this is what we have after the crucial rearranging step (5) in Agrawal's method. The idea of the Raghavarao–Nageswararao method is to start with an appropriate Youden square Y, delete a column C together with all the symbols in C and in this way go directly to the RL-form.

Construction 4.14 (Construction 1.2 in Paper III)**.** *Let Y be a Youden square, and C a column in Y. Let R be produced from Y by deleting C from Y together with all the symbols in C.*

4.3.1 Summary of paper III

We found that the 5×11 Youden square in the counterexample by Wallis and Yucas [43] never gives a triple array by this method, but from two columns we got proper double arrays. Also, we could see that there were many other Youden squares of this size for which the method works.

A $k \times v$ Youden square is a row-column representation of a $(v, k, \lambda) - SBIBD$, and we could only find one λ for which the construction never works.

Theorem 4.15 (Theorem 3.1 in Paper III)**.** *Construction 4.14 applied to Youden square Y corresponding to a projective plane of order q, that is a $(q^2 + q + 1, q + 1, 1)$-SBIBD, yields the RL form of a proper double array, regardless of the choice of column.*

From projective planes we get proper double arrays, and this is also the case when we apply Construction 4.14 to a Youden square constructed via the difference set of quadratic residues, as in Example 2.46.

Theorem 4.16 (Theorem 4.8 in Paper III)**.** *Let $q \equiv 3 \pmod 4$ be a prime power, $q > 3$. Then there exists a proper* $DA(q-1, \frac{q+1}{4}, \frac{q+1}{4}, \frac{q-3}{4} : \frac{q-1}{2} \times \frac{q+1}{2})$.

To investigate which triple arrays can be constructed this way, we can start with a given triple array, write it in RL-form R, and then try to extend R to a Youden square. Here we have a general suggestion for such an extension, but note that there are also other ways to perform it.

Construction 4.17 (Construction 3.2 in Paper III). *Let $T = TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ be a triple array with $v = r + c - 1$, and T_{RL} be the corresponding RL form. Let Y be the array constructed from T_{RL} by the following steps.*

- *Add a column C_0 to T_{RL} with r new symbols, $s_1, s_2, \ldots, s_r$, in this order (the addition step).*
- *Suppose the empty cells in column C_j in T_{RL} lie in rows $i_1, i_2, \ldots, i_{\lambda_{cc}}$. Then use the symbols $s_{i_1}, s_{i_2}, \ldots, s_{i_{\lambda_{cc}}}$ to fill the empty cells in column C_j, in such a way that no symbol is repeated in any row (the completion step).*

If the completion step can be done, then we have an extension.

Proposition 4.18 (Proposition 3.3 in Paper III). *Let T be a triple array with $v = r+c-1$ and let T_{RL} the corresponding RL form. Suppose Y is produced from T_{RL} by means of Construction 4.17, so that there is no repeated symbol in any row. Then Y is a Youden square.*

For triple arrays corresponding to biplanes, i.e., $(v, k, 2)$-SBIBDs, there is an explicit solution of how to perform the completion step.

Theorem 4.19 (Theorem 3.7 in Paper III). *Let T be a triple array with $v = r + c - 1$ and $\lambda_{cc} = 2$. Then T_{RL} can be extended to a Youden square Y that is compatible with T.*

Based on Construction 4.17 we propose the following conjecture.

Conjecture 4.20 (Conjecture 6.3 in Paper III). *Let T be a triple array with $\lambda_{cc} \geq 2$. Then there exists a Youden square Y such that Y will produce the RL form of T through Construction 4.14, for some suitably chosen column.*

The construction suggested by Raghavarao and Nageswararao [36] is closely related to Agrawal's method, but it can also be used to construct the family of Paley triple arrays.

Theorem 4.21 (Theorem 5.1 in Paper III). *Let $q \geq 5$ be an odd prime power. Then there exists a $q \times (2q+1)$ Youden square Y that via Construction 4.14 gives a triple array T for a suitable choice of column.*

The vital part of this construction is the extension $\overline{R}$.

Construction 4.22 (Construction 5.16 in Paper III). *Let R be the RL form constructed in Construction 5.12 in Paper III with parameters a, b and q. Let the array $\overline{R}$ be given by starting with an empty $q \times 2q$ array with columns labelled by $\{w_0, w_1, \ldots, w_{q-1}, w'_0, w'_1, \ldots, w'_{q-1}\}$ in this order and rows labelled by $\{w_0, w_1, \ldots, w_{q-1}\}$ in this order and performing the following steps:*

- *adding the column $\infty = (w'_0 = 0', w'_1, \ldots, w'_{q-1})$ as the first column,*
- *for columns $\{w_0, w_1, \ldots, w_{q-1}\}$, if $a \in Q$ and $(w_i - w_s) \in N$, or if $a \in N$ and $(w_i - w_s) \in Q$, setting*

$$\overline{R}(w_i, w_s) = (\alpha(w_s - w_i) + w_i)',$$

- *for columns* $\{w'_0, w'_1, \ldots, w'_{q-1}\}$*, if* $b \in Q$ *and* $(w_s - w_i) \in Q$*, or if* $b \in N$ *and* $(w_s - w_i) \in N$*, setting*

$$\overline{R}(w_i, w'_s) = (\beta(w_i - w_s) + w_i)',$$

where $\alpha, \beta \in GF(q)^*$ *are chosen as follows. Suppose* $q \equiv 1 \pmod 4$*, then choose* α *and* β *such that* $\alpha - 1, \beta - 1 \in Q$ *and such that* α *and* β *are in the same set* Q *or* N*. Suppose* $q \equiv 3 \pmod 4$*, then we have two cases:*

- ⋆ *if* a *and* b *are in the same set* Q *or* N*, then choose* α *and* β *such that* $\alpha - 1$ *and* $\beta - 1$ *are both in one of the sets* Q *and* N *but* α *and* β *are in different sets of* Q *and* N*,*
- ⋆ *if* a *and* b *are in different sets of* Q *and* N*, then choose* α *and* β *such that* $\alpha-1, \beta-1 \in N$ *and such that* α *and* β *are in the same set* Q *or* N*.*

Then the RL-form R and the extension $\overline{R}$ are superimposed to form a Youden square Y. In Table 4.2 we have extended Table 3.2 for Paley triple arrays to also comprise the choises for α in Construction 4.22.

Type	Description
$T_{1,1}$:	$q \equiv 1 \pmod 4$ and $a \in Q$,
$T_{1,2}$:	$q \equiv 1 \pmod 4$ and $a \in N$,
$T_{3,1,1}$:	$q \equiv 3 \pmod 4$, $a, b \in Q$, $(a-1) \in Q$ and $(\alpha - 1) \in Q$,
$T_{3,1,2}$:	$q \equiv 3 \pmod 4$, $a, b \in Q$, $(a-1) \in Q$ and $(\alpha - 1) \in N$,
$T_{3,2,1}$:	$q \equiv 3 \pmod 4$, $a, b \in Q$, $(a-1) \in N$ and $(\alpha - 1) \in Q$,
$T_{3,2,2}$:	$q \equiv 3 \pmod 4$, $a, b \in Q$, $(a-1) \in N$ and $(\alpha - 1) \in N$,
$T_{3,3,1}$:	$q \equiv 3 \pmod 4$, $a, b \in N$, $(a-1) \in Q$ and $(\alpha - 1) \in Q$,
$T_{3,3,2}$:	$q \equiv 3 \pmod 4$, $a, b \in N$, $(a-1) \in Q$ and $(\alpha - 1) \in N$,
$T_{3,4,1}$:	$q \equiv 3 \pmod 4$, $a, b \in N$, $(a-1) \in N$ and $(\alpha - 1) \in Q$,
$T_{3,4,2}$:	$q \equiv 3 \pmod 4$, $a, b \in N$, $(a-1) \in N$ and $(\alpha - 1) \in N$,
$T_{3,5}$:	$q \equiv 3 \pmod 4$, $a \in Q$ and $b \in N$,
$T_{3,6}$:	$q \equiv 3 \pmod 4$, $a \in N$ and $b \in N$.

Table 4.2: Twelve types of Youden squares that give Paley triple arrays. Note that the conditions here are minimal for identifying a type and imply further conditions by Theorems 3.35 and 3.36 and Construction 4.22.

Example 4.23. *Let us construct a* 9×19 *Youden square that via Construction 1.2 in Paper III gives a* $TA(18, 5, 5, 4, 5 : 9 \times 10)$*. Like in Example 3.37, we construct* $GF(9)$ *by letting* γ *be a root of the primitive polynomial* $x^2 + x + 2$ *over* $GF(3)$*, so* γ *is a primitive element of* $GF(9)$ *that generates the multiplicative group of* $GF(9)$*. We construct a* $q \times 2q$ *array* R *by labeling the rows* $\{0, \gamma, \gamma^2, \ldots, \gamma^{q-1}\}$ *and the columns by* $\{0, \gamma, \gamma^2, \ldots, \gamma^{q-1}, 0', \gamma', \gamma^{2'}, \ldots, \gamma^{q-1'}\}$ *and allocate entries to it by Construction 5.12 of Paper III. For this we choose* $a = \gamma^4$ *and* $b = \gamma^6$ *which satisfy the conditions of Corollary 5.14 in Paper III, so* R *will be the RL-form of a triple array.*

To exemplify the construction of R *we compute one entry of* R*, let us do* $R(0, \gamma^2)$*. As* $0 - \gamma^2 = \gamma + 2 = \gamma^6 \in Q$ *we have*

$$R(0, \gamma^2) = \gamma^4(\gamma^2 - 0) + 0 = \gamma^6.$$

Then we construct the extension $\overline{R}$ *by Construction 5.16 in Paper III. For this we need to choose* α *and* β*, and in this case the only conditions are that* $\alpha - 1, \beta - 1 \in Q$*, and that* α *and* β *are in the same set* Q *or* N*. We choose* $\alpha = \beta = \gamma^4$ *and compute* $\overline{R}(\gamma^6, \gamma^4)$ *to exemplify. As* $\gamma^4 - \gamma^6 = 2 - (\gamma + 2) = 2\gamma = \gamma^5 \in N$ *we have*

$$\overline{R}(\gamma^6, \gamma^4) = (\gamma^4\gamma^5 + \gamma^6)' = (2\gamma + 2)' = \gamma^{3'}.$$

Finally, we superimpose R and $\overline{R}$, which includes that we add a first column $\infty = (0', \gamma', \ldots, \gamma^{8'})$ to get the $q \times 2q + 1$ Youden square Y here below. Applying Construction 1.2 in Paper III to Y modulo column ∞ gives the RL-form of the Paley triple array of type $T_{1,1}$ in Example 3.37.

$0'$	∞	$\gamma^{5'}$	γ^6	$\gamma^{7'}$	γ^8	γ'	γ^2	$\gamma^{3'}$	γ^4	0	γ^3	$\gamma^{6'}$	γ^5	$\gamma^{8'}$	γ^7	$\gamma^{2'}$	γ	$\gamma^{4'}$
γ'	$\gamma^{5'}$	∞	$\gamma^{4'}$	γ^8	$\gamma^{2'}$	$0'$	γ^7	γ^6	γ^3	γ^2	γ	γ^5	$\gamma^{8'}$	0	γ^4	$\gamma^{7'}$	$\gamma^{6'}$	$\gamma^{3'}$
$\gamma^{2'}$	γ^6	$\gamma^{4'}$	∞	γ^5	γ'	γ^3	0	$\gamma^{8'}$	$\gamma^{7'}$	$\gamma^{6'}$	γ^7	γ^2	$\gamma^{5'}$	γ^8	$\gamma^{3'}$	$0'$	γ^4	γ
$\gamma^{3'}$	$\gamma^{7'}$	γ^8	γ^5	∞	$\gamma^{6'}$	γ^2	$\gamma^{4'}$	$0'$	γ	γ^4	$\gamma^{8'}$	$\gamma^{5'}$	γ^3	γ^7	$\gamma^{2'}$	0	γ^6	$\gamma^{8'}$
$\gamma^{4'}$	γ^8	$\gamma^{2'}$	γ'	$\gamma^{6'}$	∞	γ^7	$\gamma^{3'}$	γ^5	0	$\gamma^{8'}$	γ^6	γ^3	γ	γ^4	$\gamma^{7'}$	γ^2	$\gamma^{5'}$	$0'$
$\gamma^{5'}$	γ'	$0'$	γ^3	γ^2	γ^7	∞	$\gamma^{8'}$	γ^4	$\gamma^{6'}$	γ^6	γ^8	$\gamma^{3'}$	$\gamma^{2'}$	$\gamma^{7'}$	γ^5	γ	$\gamma^{4'}$	0
$\gamma^{6'}$	γ^2	γ^7	0	$\gamma^{4'}$	$\gamma^{3'}$	$\gamma^{8'}$	∞	γ	$\gamma^{5'}$	$\gamma^{2'}$	$\gamma^{7'}$	$0'$	γ^8	γ^5	γ^3	γ^6	γ'	γ^4
$\gamma^{7'}$	$\gamma^{3'}$	γ^6	$\gamma^{8'}$	$0'$	γ^5	γ^4	γ	∞	$\gamma^{2'}$	γ^8	$\gamma^{6'}$	0	γ^2	$\gamma^{5'}$	$\gamma^{4'}$	γ'	γ^7	γ^3
$\gamma^{8'}$	γ^4	γ^3	$\gamma^{7'}$	γ	0	$\gamma^{6'}$	$\gamma^{5'}$	$\gamma^{2'}$	∞	$\gamma^{4'}$	$\gamma^{3'}$	γ^6	γ'	$0'$	γ^2	γ^7	γ^5	γ^8

Author's contributions to this paper: This paper was a collaborative effort, and all ideas and results were discussed with my co-author. The construction of Paley triple arrays as well as the result on projective planes and difference sets were initiated and carried out by me.

Part II

Solitons

Chapter 5

Introduction to an operator theoretic approach to soliton theory

> *"True laws of Nature cannot be linear."*
> Albert Einstein

5.1 Historical background

In 1834, when John Scott Russel was trying to find the most suitable design for canal boats, he made experiments at the Union Canal near Edinburgh. One day when he was sitting on his horse close to the canal, he happened to catch sight of a peculiar wave phenomenon.

> *"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminuation of speed. I followed it on hourseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its heights gradually diminished, and after a chase of one or two miles I lost it in the windlings of the channel."*
> - J. S. Russel, "Reports on Waves", (14th Meeting of the British Association for the Advancement of Science, 1844)

Certainly, Russel was not the first to observe such a wave in nature, but he seems to be the first to draw the interest of a larger scientific audience to the phenomenon of these *solitary waves* or *wave of translation* as he also called them. As a solid scientist and engineer, he built a long wave tank in his back garden where he was able to reproduce such waves in experiments. He guessed a mathematical description of their shape, which was confirmed by much later achievements in the topic. Actually it takes quite a while, until Korteweg and de Vries [69] derive the differential equation governing the dynamics of shallow water waves. Despite this

Figure 5.1: Simulation of J. S. Russel's observation (Heriot–Watt University 1995).

important conceptual progress, the start of modern soliton theory is usually located at the celebrated Fermi-Pasta-Ulam experiments [61]. Their surprising outcome strikingly underlines that stability phenomena, such as the existence of localized shape-preserving waves, should be of principle importance in mathematical physics. Shortly afterwards, the famous inverse scattering method is developed by Gardner, Green, Kruskal and Miura [63]. From that moment on, the topic of integrable systems has been a very active research area with quick progress and many brilliant discoveries linking the topic to several major parts of mathematics like differential geometry, algebraic geometry and topology, representation theory and operator theory. In parallel, physicists continue to discover applications in disciplines as varied as quantum optics, general relativity, crystallography, biophysics (see for example [62] and references therein).

5.2 The operator method

The present work is based on an operator method for the study of soliton equations which can be described as follows:

1. The starting point is a the soliton equation one is interested in together with a solution u of this equation. Typically this solution is the 1-soliton solution, which depends on a parameter $a \in \mathbb{C}$.

2. The first step consists in finding a noncommutative counterpart of the given soliton equation. More precisely, one looks for a differential equation for functions with values in some suitable (not necessarily commutative) algebra $\mathcal{A}$ which reduces to the original equation if $\mathcal{A} = \mathbb{C}$.

   Simultaneously one looks for a noncommutative counterpart U of u such that U solves the noncommutative equation. Typically, in this process the scalar parameter a is blown up to a parameter $A \in \mathcal{A}$.

3. The second step is to construct a solution formula for the original equation by applying a suitable functional $\tau : \mathcal{A} \mapsto \mathbb{C}$ to U. The crucial point is to do it in such a way that the solution property is maintained.

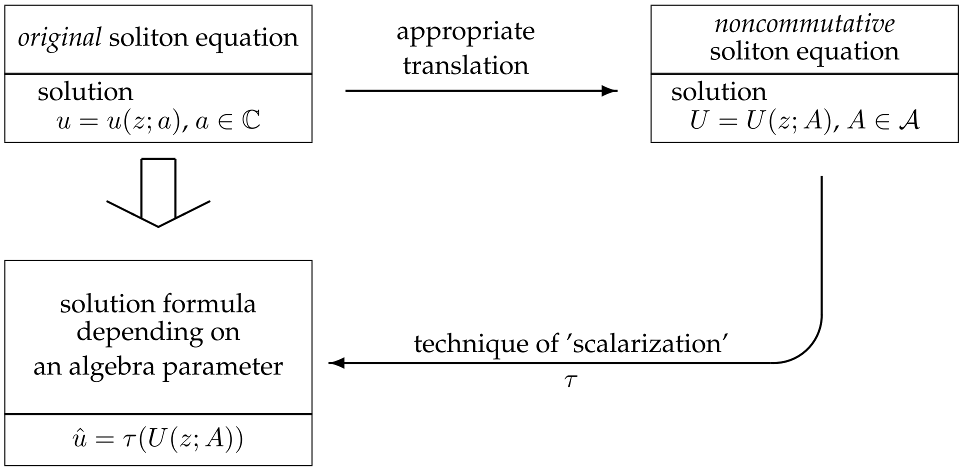


Figure 5.2: The operator method.

The main advantage of this method is that it provides us with a solution formula depending on an algebra parameter A. Originally the operator method stems from pioneering work of Marchenko [71], who pursued it using as algebra $\mathcal{A} = \mathcal{L}(H)$, the algebra of bounded linear operators on a Hilbert space H. Later it was generalized by B. Carl who placed the method in the context of modern Banach space geometry. Building on his seminal work [48] on the Korteweg-de Vries equation, it was then amply developed by several authors, with contributions to the following aspects:

- continuous soliton equations [57], [58], [82], [83],
- lattices [51], [57], [58], [78], [81],
- hierarchies and recursion operators [53], [54], [55], [87],
- ZS and AKNS systems [52], [84], [86],
- vector- and matrix-soliton equations [67], [83], [84], [86],
- application to the initial value problem [51], [52], [58],
- semigroup techniques [56], [58], [67],
- multiple pole solutions [79], [80], [85],
- countable nonlinear superposition [57], [78], [81], [82], [86].

The subsequent text is organized as follows: In Section 5.3 we illustrate the operator method for the prototypical case of the Korteweg-de Vries equation. To make our argumentation as transparent as possible, we focus on the case $\mathcal{A} = M_{N,N}(\mathbb{C})$, the algebra of $N \times N$-matrices over $\mathbb{C}$. As a first application of the method, N-soliton solutions are derived in Section 5.4 and the principle of nonlinear superposition is discussed. Then background material for the generalization to the algebra of bounded operators on some Banach space is provided in Sections 5.6 and 5.7 following closely [57] in the exposition of the material. In Section5.5 an brief introduction to the famous direct approach originating from R. Hirota is given. Finally the main results of the present thesis are presented and put into the context of the recent literature in Section 5.8.

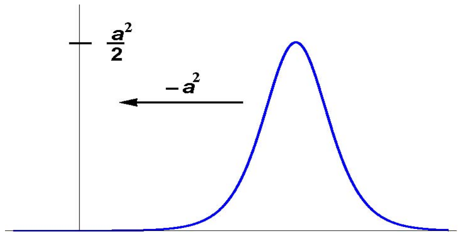


Figure 5.3: Snapshot of the 1-soliton solution of the Korteweg-de Vries equation.

5.3 Illustration of the method for the Korteweg-de Vries equation

In this section we implement the operator method in the prototypical case of the Korteweg-de Vries equation,

$$u_t = u_{xxx} + 6uu_x. \tag{5.1}$$

The results are due to Aden and Carl [48] who implemented the method for the first time for the potential Korteweg-de Vries equation, see also [57] for a presentation for the Korteweg-de Vries equation itself.

The 1-soliton solution of the Korteweg-de Vries equation reads

$$u(x,t) = \frac{a^2}{2}\cosh^{-2}\Big(\frac{a}{2}(x + a^2 t + \varphi)\Big). \tag{5.2}$$

Note that the solution depends on two real parameters a, φ, the first characterizing its height and velocity, the second its initial position.

5.3.1 The noncommutative Korteweg-de Vries equation and the noncommutative analogue of its soliton solution

The noncommutative Korteweg-de Vries equation reads

$$U_t = U_{xxx} + 3(UU_x + U_xU), \tag{5.3}$$

where U is a function with values in the $N \times N$-matrices over $\mathbb{C}$. Note that this is a very canonical noncommutative interpretation of (5.1) where noncommutativity is reflected by replacing the nonlinear term in a symmetric way.

The following reformulation of (5.2) motivates the noncommutative interpretation of the soliton solution. Setting $\ell(x,t) = \exp(ax + a^3 t)b$, where $b = \exp(a\varphi)$, one finds

$$\begin{aligned} u(x,t) &= 2a^2\left(\sqrt{\ell(x,t)} + \frac{1}{\sqrt{\ell(x,t)}}\right)^{-2} \\ &= 2a^2\,(1+\ell(x,t))^{-2}\,\ell(x,t) \\ &= \frac{\partial}{\partial x}\Big((1+\ell(x,t))^{-1}\,2a\ell(x,t)\Big). \end{aligned}$$

Hence, a reasonable noncommutative counterpart is

$$U(x,t) = \frac{\partial}{\partial x}\Big((I + L(x,t))^{-1}\,\big(AL(x,t) + L(x,t)A\big)\Big), \tag{5.4}$$

where $L(x,t) = \exp(Ax + A^3t)B$ and A, B are constant $N \times N$-matrices. Of course, in the matrix case invertibility of $I + L(x,t)$ is no longer a priori guaranteed.

The next proposition, which is a corollary of [48, Theorem III B 2 (i)], see also [57, Proposition 5.1], states the solution property of (5.4).

Proposition 5.1. *Let A, B be $N \times N$-matrices and define $L(x,t) = \exp(Ax + A^3t)B$. Then (5.4) is a solution of the noncommutative Korteweg-de Vries equation (5.3) on $\Omega = \{(x,t) \in \mathbb{R}^2 \mid \det\big(I + L(x,t)\big) \neq 0\}$.*

Proof. Let $V = (I+L)^{-1}(AL + LA)$. Using $L_x = AL$ and the noncommutative derivation rule for inverses $\big((I_L)^{-1}\big) = -(I+L)^{-1}L_x(I+L)^{-1}$ we obtain

$$\begin{aligned} V_x &= -(I+L)^{-1}L_x(I+L)^{-1}(AL+LA) + (I+L)^{-1}(AL_x + L_xA) \\ &= -(I+L)^{-1}AL(I+L)^{-1}(AL+LA) + (I+L)^{-1}A(AL+LA) \\ &= (I+L)^{-1}A(I+L)^{-1}(AL+LA) \\ &= (I+L)^{-1}AV, \end{aligned}$$

where we have used the identity $L(I+L)^{-1} = I - (I+L)^{-1}$ to replace the term $L(I+L)^{-1}$ in the above calculation. Similarly,

$$\begin{aligned} V_{xx} &= 2(I+L)^{-1}A(I+L)^{-1}AV - VA^2V, \\ V_{xxx} &= 6(I+L)^{-1}A(I+L)^{-1}A(I+L)^{-1}AV \\ &\quad -3(I+L)^{-1}A^2(I+L)^{-1}AV - 3(I+L)^{-1}A(I+L)^{-1}A^2V \\ &\quad +(I+L)^{-1}A^3V \end{aligned}$$

and

$$V_t = (I+L)^{-1}A^3(I+L)^{-1}(AL+LA).$$

On the other hand,

$$\begin{aligned} V_x^2 &= (I+L)^{-1}A(I+L)^{-1}(AL+LA)(I+L)^{-1}AV \\ &= (I+L)^{-1}A(I+L)^{-1}\Big(A(I+L) + (I+L)A - 2A\Big)(I+L)^{-1}AV \\ &= (I+L)^{-1}A(I+L)^{-1}A^2V + (I+L)^{-1}A^2(I+L)^{-1}AV \\ &\quad -(I+L)^{-1}A(I+L)^{-1}A(I+L)^{-1}AV. \end{aligned}$$

In summary this shows that V is a solution of the noncommutative potential Korteweg-de Vries equation

$$V_t = V_{xxx} + 3V_x^2.$$

From this we infer that $V_{tx} = V_{xxxx} + 3(V_{xx}V_x + V_xV_{xx})$ using the noncommutative product rule. In other words $U = V_x$ solves (5.3). □

5.3.2 Derivation of a solution formula for the scalar Korteweg-de Vries equation

To derive a solution formula for the Korteweg-de Vries equation (5.1), the main idea is to apply a (linear) functional $\tau : \mathcal{M}_{N,N}(\mathbb{C}) \mapsto \mathbb{C}$ to a solution of the noncommutative Korteweg-de Vries equation (5.3). Of course the key point is to do this in such a way that the solution property is maintained.

As a motivation we start from the observation that, if $U(x,t)$ is a solution of the noncommutative Korteweg-de Vries equation (5.3), then so is $T^{-1}U(x,t)T$ for every constant invertible matrix T. In other words, conjugation with a constant matrix does not change the solution property. Thus it is reasonable to look for a functional τ that maps $U(x,t)$ and $T^{-1}U(x,t)T$ to the same scalar function $u(x,t)$. The next lemma shows that this restricts the choice of the functional considerably.

Lemma 5.2. *Let $\tau : \mathcal{M}_{N,N}(\mathbb{C}) \mapsto \mathbb{C}$ be a linear functional with $\tau(ST) = \tau(TS)$ for all $N \times N$-matrices S, T. Then τ is a multiple of the trace* tr.

Proof. Let the function τ be given by $\tau(T) = \sum_{i,j=1}^{N} \lambda_{ij} t_{ij}$ for $T = (t_{ij})_{i,j=1}^{N}$. Denote by E_{ij} the matrix with the only non-vanishing entry 1 in the ith row and jth column. Then $\lambda_{ij} = \tau(E_{ij}) = \tau(E_{ii}E_{ij}) = \tau(E_{ij}E_{ii}) = \tau(\delta_{ij}E_{ii}) = \delta_{ij} = 0$ for $j \neq i$ and $\lambda_{jj} = \tau(E_{jj}) = \tau(E_{j1}E_{1j}) = \tau(E_{1j}E_{j1}) = \tau(E_{11}) = \lambda_{11}$ for all j, Hence $\tau(T) = \lambda_{11} \sum_{j=1}^{N} t_{jj} = \lambda_{11}\text{tr}(T)$. □

In the case $N = 1$, the fact that τ maintains the solution property necessarily means that τ is the identity map. In view of the above motivation, the right generalization for $N > 1$ appears to be the trace tr.

The difficulty is that in order to maintain the solution property, τ needs to be multiplicative at least to a certain extent. However, since the trace is not multiplicative on all of $\mathcal{M}_{N,N}(\mathbb{C})$, we need to restrict the class of matrices.

To this end we need some notation. A matrix T is called *one-dimensional*, if its range ran$(T) = \{Tv \mid v \in \mathbb{C}^N\}$ is a one-dimensional subspace of $\mathbb{C}^N$. Note that for ran$(T) =$ span$\{c\}$ we have $T = ca^t$ where the entries of the vector a are given by $Te_j = a_j c$. Since $\ker(T) = \{v \in \mathbb{C}^N \mid \langle a, v\rangle = 0\}$, the kernel of T is determined by a. Vice versa, two one-dimensional matrices S, T with the same kernel can be written in the form $S = ca^t$, $T = da^t$ with the same vector a. Finally we observe that for $T = ca^t$ the trace can be calculated by the simple formula tr$(T) = a^t c$.

Lemma 5.3. *On the set of one-dimensional matrices with the same kernel, the trace is multiplicative.*

Proof. Let $S = ca^t$, $T = da^t$. Then $ST = (ca^t)(da^t) = c(a^t d)a^t = \text{tr}(T)ca^t = \text{tr}(T)S$. Hence, $\text{tr}(ST) = \text{tr}\big(\text{tr}(T)S\big) = \text{tr}(S)\text{tr}(T)$. □

The following proposition ([48, Proposition III B 1], see also [57, Proposition 5.2]) is a summary of the above discussion.

Proposition 5.4. *If $U = U(x,t)$ is a solution of the noncommutative Korteweg-de Vries equation (5.3) which is one-dimensional with constant kernel, then $u = \text{tr}(U)$ is a solution of the scalar Korteweg-de Vries equation (5.1).*

Proof. By assumption U can be written in the form $U(x,t) = c(x,t)a^t$ with some constant vector a and a vector function $c = c(x,t)$. But then $U_x(x,t) = c_x(x,t)a^t$, which means that U, U_x are one-dimensional with the same kernel. By Lemma 5.3, $\text{tr}(UU_x) = \text{tr}(U_xU) = \text{tr}(U)\text{tr}(U_x)$. As a linear functional on $\mathcal{M}_{N,N}(\mathbb{C})$, the trace is continuous. Thus $\text{tr}(U_x) = \text{tr}(U)_x$. Together, this shows

$$\begin{aligned} u_t &= \text{tr}(U)_t = \text{tr}(U_t) \overset{(5.3)}{=} \text{tr}\Big(U_{xxx} + 3(UU_x + U_xU)\Big) \\ &= \text{tr}(U_{xxx}) + 3\Big(\text{tr}(UU_x) + \text{tr}(U_xU)\Big) = \text{tr}(U)_{xxx} + 6\,\text{tr}(U)\text{tr}(U)_x \\ &= u_{xxx} + 6uu_x. \end{aligned}$$

□

Applying Proposition 5.4 to the concrete solution (5.4) yields the desired solution formula. This is the contents of the next theorem. See [48, Theorem III B 2 (ii)], [57, Theorem 5.4] for a corresponding solution formula in the more general context of quasi-Banach ideals.

Theorem 1. *Let A, B be $N \times N$-matrices such that $AB + BA$ is one-dimensional, and define $L(x,t) = \exp(Ax + A^3 t)B$. Then*

$$u(x,t) = 2\frac{\partial^2}{\partial x^2} \log\det\big(I + L(x,t)\big)$$

is a solution of the Korteweg-de Vries equation (5.1) on $\Omega = \{(x,t) \in \mathbb{R}^2 \mid \det\big(I + L(x,t)\big) \neq 0\}$.

Proof. For $AB + BA = ca^t$, it is straightforward to verify that the solution (5.4) has the form $U(x,t) = d_x(x,t)a^t$ with the vector-function

$$d(x,t) = \big(I + \exp(Ax + A^3 t)B\big)^{-1} \exp(Ax + A^3 t)c.$$

Thus the assumptions of Proposition 5.4 are satisfied and we get a solution u of the Korteweg-de Vries equation (5.1) by taking the trace of U. On the other hand, from the representation (5.4), we find

$$\begin{aligned} u &= \mathrm{tr}(U) \\ &= \frac{\partial}{\partial x}\mathrm{tr}\Big((I+L)^{-1}(AL+LA)\Big) \\ &= \frac{\partial}{\partial x}\Big(\mathrm{tr}\big((I+L)^{-1}AL\big) + \mathrm{tr}\big((I+L)^{-1}LA\big)\Big). \end{aligned}$$

Since L and $(I+L)^{-1}$ commute, we observe $\mathrm{tr}\big((I+L)^{-1}LA\big) = \mathrm{tr}\big(L(I+L)^{-1}A\big)$, and, using the trace property, then $\mathrm{tr}\big(L\cdot(I+L)^{-1}A\big) = \mathrm{tr}\big((I+L)^{-1}A\cdot L\big)$. Hence,

$$\begin{aligned} u &= 2\frac{\partial}{\partial x}\mathrm{tr}\big((I+L)^{-1}AL\big) \\ &= 2\frac{\partial}{\partial x}\mathrm{tr}\big((I+L)^{-1}L_x\big). \end{aligned}$$

To complete the proof, it remains to apply the subsequent lemma. □

Lemma 5.5. *Let $T = T(x)$ be a matrix-function which is differentiable and invertible in some open interval J in $\mathbb{R}$. Then, for all $x \in J$, it holds*

$$\Big(\det\big(T(x)\big)\Big)^{-1}\frac{d}{dx}\det\big(T(x)\big) = \mathrm{tr}\Big(T^{-1}(x)\frac{d}{dx}T(x)\Big).$$

Proof. Fix $x_0 \in J$ and set $S(x) = T^{-1}(x_0)T(x)$. Note that then $S(x_0) = I_N$. Writing $S(x) = (S^{(1)}(x), \dots, S^{(N)}(x))$ where $S^{(j)}(x)$ is the jth column of $S(x)$, we thus have $S^{(j)}(x_0) = e_j$, the jth standard unit vector, and

$$\begin{aligned} &\big(\det(T(x_0))\big)^{-1}\frac{d}{dx}\Big|_{x=x_0}\det\big(T(x)\big) = \\ &= \frac{d}{dx}\Big|_{x=x_0}\det\big(S(x)\big) \\ &= \frac{d}{dx}\Big|_{x=x_0}\det\big(S^{(1)}(x), \dots, S^{(N)}(x)\big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N} \det\Big(S^{(1)}(x_0),\ldots,S^{(j-1)}(x_0),\frac{d}{dx}\Big|_{x=x_0} S^{(j)}(x),S^{(j+1)}(x_0),\ldots,S^{(N)}(x_0)\Big)\\
&= \sum_{j=1}^{N} \det\Big(e_1,\ldots,e_{j-1},\frac{d}{dx}\Big|_{x=x_0} S^{(j)}(x),e_{j+1},\ldots,e_N\Big)\\
&= \mathrm{tr}\Big(\frac{d}{dx}\Big|_{x=x_0} S(x)\Big)\\
&= \mathrm{tr}\Big(T^{-1}(x_0)\frac{d}{dx}T(x_0)\Big).
\end{aligned}
$$

□

5.4 N-soliton solutions and nonlinear superposition

In this section we rederive the well-known N-soliton solutions of Hirota [65] by inserting diagonal matrices A into the solution formula in Proposition 1 and study their collision behavior for $N = 2$ and $N = 3$.

Let

$$
A = \begin{pmatrix} k_1 & & 0 \\ & \ddots & \\ 0 & & k_N \end{pmatrix}.
$$

where $k_j \in \mathbb{C}$ are pairwise different and $k_i + k_j \neq 0$ for all $i, j = 1, \ldots, N$.

In order to satisfy the one-dimensionality condition of Theorem 1, we consider the Sylvester equation

$$
AX + XA = C \tag{5.5}
$$

for a given right-hand side C. Of course we are mainly interested in the case that C is one-dimensional. However, it is well-known that (5.13) is uniquely solvable if and only if $0 \notin \mathrm{spec}(A) + \mathrm{spec}(A)$ (see for example [50] for a survey on Sylvester's equation in the matrix case).

In our situation, $\mathrm{spec}(A) + \mathrm{spec}(A) = \{k_i + k_j \mid i, j = 1, \ldots, N\}$ showing that (5.13) is always solvable. Indeed, it is straightforward to verify that, for $a, c \in \mathbb{C}^N$ arbitrary, the unique solution B of

$$
AB + BA = ca^t
$$

is given by

$$
B = \left(\frac{a_j c_i}{k_i + k_j}\right)_{i,j=1}^{N}.
$$

Thus, according to Theorem 1, a solution of the Korteweg-de Vries equation is given by $u(x,t) = 2\partial_x^2 \log p(x,t)$, where

$$
\begin{aligned}
p(x,t) &= \det\left(\left(\delta_{ij} + \exp\left(k_i x + k_i^3 t\right)\frac{a_j c_i}{k_i + k_j}\right)_{i,j=1}^{N}\right)\\
&= 1 + \sum_{m=1}^{N}\sum_{i_1<\ldots<i_m} \det\left(\left(\exp\Big(k_{i_j}x + k_{i_j}^3 t\Big)\frac{a_{i_{j'}} c_{i_j}}{k_{i_j} + k_{i_{j'}}}\right)_{j,j'=1}^{m}\right)\\
&= 1 + \sum_{m=1}^{N}\sum_{i_1<\ldots<i_m} \prod_{j=1}^{m} a_{i_j} c_{i_j} \exp\Big(k_{i_j}x + k_{i_j}^3 t\Big) \det\left(\left(\frac{1}{k_{i_j} + k_{i_{j'}}}\right)_{j,j'=1}^{m}\right)
\end{aligned}
$$

$$= \quad 1+\sum_{m=1}^{N}\sum_{i_1<\ldots<i_m}\prod_{j=1}^{m}\frac{a_{i_j}c_{i_j}}{2k_{i_j}}\exp\left(k_{i_j}x+k_{i_j}^3t\right)\prod_{j'=j+1}^{m}\left(\frac{k_{i_j}-k_{i_{j'}}}{k_{i_j}+k_{i_{j'}}}\right)^2$$

where we refer to [75] for the identity

$$\det\left(\left(\frac{1}{k_i+k_j}\right)_{i,j=1}^{m}\right)=\prod_{i=1}^{m}\frac{1}{2k_i}\prod_{j=i+1}^{m}\left(\frac{k_i-k_j}{k_i+k_j}\right)^2$$

which has been used in the last step.

Abbreviating $A_{ij}=\big((k_i-k_j)/(k_i+k_j)\big)^2$ and $\ell_j(x,t)=\exp(k_jx+k_j^3t+\delta_j)$, where δ_j is defined by $\exp(\delta_j)=(a_jc_j)/(2k_j)$, we find

$$p(x,t)=1+\sum_{m=1}^{N}\sum_{i_1<\ldots<i_m}\prod_{j=1}^{m}\ell_{i_j}(x,t)\prod_{j'=j+1}^{m}A_{i_ji_{j'}}.$$

For $0<k_1<\ldots<k_N$ and $\delta_j\in\mathbb{R}$ this solution is the famous N-soliton solution of Hirota.

Let us take a closer look at the 2-soliton solution. As shown in Figure 5.4, for large negative times the solution consists of two single solitons which are well separated and travel each with constant velocity. As time goes by, the faster soliton catches up with the smaller one and overtakes it. The resulting collision does neither change the shape nor the velocity of the solitons, its only effect is that the both solitons experience a phase shift: The faster one is shifted forward, the slower one backward. For large positive times the picture is the same as in the beginning except that the solitons appear in reversed order.

In the generic situation where one soliton is much larger than the other, a single maximum is formed during collision. However, if the difference of height between the solitons is small there are always two maxima present, see Figure 5.5. For more details we refer to [49] and references therein.

We conclude the section with an illustration of the 3-soliton solution in Figure 5.6.

5.5 Hirota's bilinear method

In this section we explain the connection between the operator method and the celebrated bilinear method of Hirota, a direct method for constructing N-soliton solutions. Introduced by Hirota [65] for the Korteweg-de Vries equation, it has since become one of the most powerful direct methods in soliton theory. For a survey on the state of art we refer to [66], see also [72].

To start with, we introduce the D-operator (or *Hirota derivative*), a binary differential operator acting on a pair of functions $a(x,t)$, $b(x,t)$ which is defined by

$$D_x^mD_t^n(a,b)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n a(x,t)b(x',t')\Bigg|_{\substack{x=x'\\t=t'}}.$$

As an illustration, we calculate $D_xD_t(a,b)$. Since

$$\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)a(x,t)b(x',t')$$

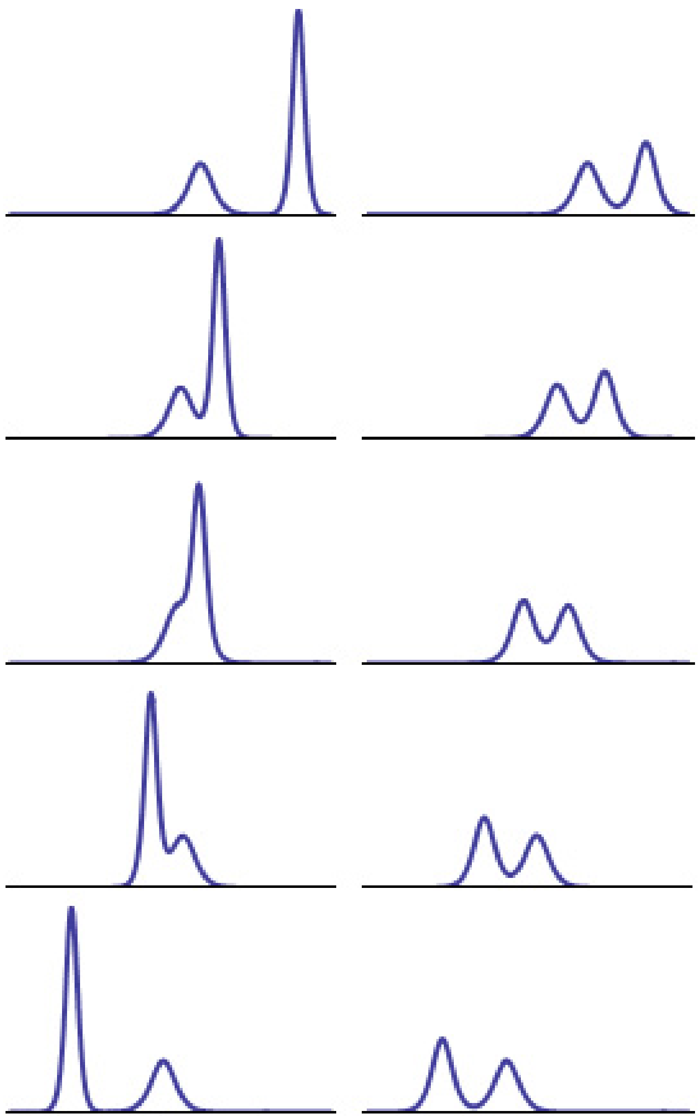


Figure 5.4: Two typical interaction patterns for the 2-soliton solution. On the left we have chosen $k_1 = 0.5$, $k_2 = 1$, on the right $k_1 = 0.5$, $k_2 = 0.7$. In both cases the solution is shown at succeeding times.

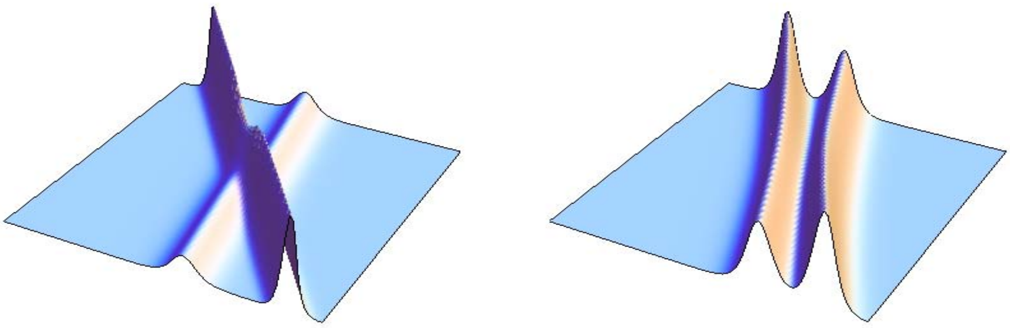


Figure 5.5: The same 2-soliton solutions as in Figure 5.4 depicted over the xt-plane.

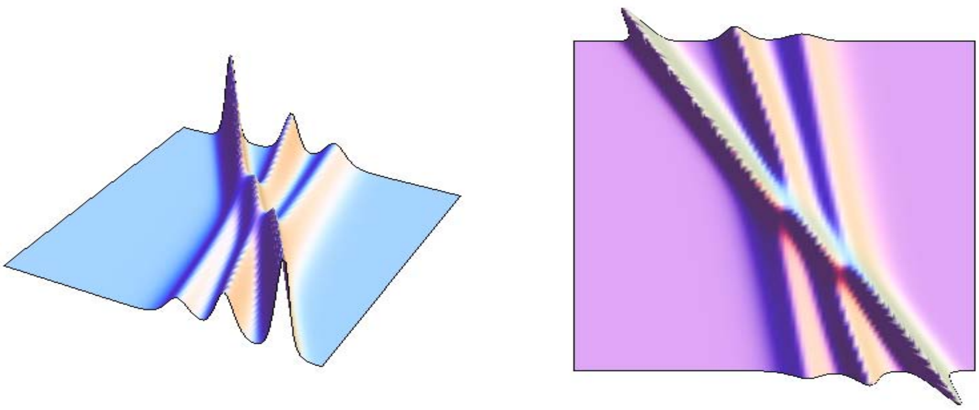


Figure 5.6: 3-soliton solution for $k_1 = 0.5$, $k_2 = 0.7$ and $k_3 = 1$ shown over the xt-plane.

$$
\begin{aligned}
&= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)\Big(a_t(x,t)b(x',t') - a(x,t)b_{t'}(x',t')\Big) \\
&= a_{tx}(x,t)b(x',t') - a_x(x,t)b_{t'}(x',t') - a_t(x,t)b_{x'}(x',t') + a(x,t)b_{t'x'}(x',t'),
\end{aligned}
$$

we get

$$D_x D_t(a,b) = \big(a_{xt}b + ab_{xt}\big) - \big(a_x b_t + a_t b_x\big) \tag{5.6}$$

For a more thorough discussion of the D-operators properties see [66], [72].

Hirota showed that in many cases a soliton equation can be rewritten, via an appropriate transformation of variables, in terms of the D-operator. This process is called bilinearization.

For the Korteweg-de Vries equation, substitution of the logarithmic transformation

$$u = 2\frac{\partial^2}{\partial x^2}\log f \tag{5.7}$$

into (5.1), one obtains after one integration in x (the integration constant denoted by c) and a little manipulation

$$ff_{xt} - f_x f_t = ff_{xxxx} - 4f_x f_{xxx} + 3f_{xx}^2 + cf^2.$$

In view of (5.6), the left-hand side of the above equation coincides with $D_x D_t(f,f)$ up to a factor $1/2$. Similarly, it is straightforward to express also its the right-hand side in terms of the D-operator. As a result,

$$D_x D_t(f,f) = D_x^4(f,f) + cf^2,$$

and taking into account the boundary conditions for obtaining soliton solutions, $f_x, f_{xx}, \ldots \to 0$ for $x \to \pm\infty$, we arrive at the bilinear form of the Korteweg-de Vries equation

$$D_x\big(D_t - D_x^3\big)(f,f) = 0. \tag{5.8}$$

Vice versa, if f solves (5.8), a solution u of the Korteweg-de Vries equation is obtained by (5.7).

To solve (5.8), one uses a formal perturbation expansion ansatz. Inserting the expression

$$f(x,t) = \sum_{n=0}^{\infty} \epsilon^n f_n(x,t) \tag{5.9}$$

with $f_0(x,t) \equiv 1$, and comparing coefficients in the powers of ϵ, (5.8) is turned into a hierarchy of equations for $f_n(x,t)$ ($n \geq 1$), which can be solved successively.

In particular, starting from

$$f_1(x,t) = \sum_{j=1}^{N} \exp(\eta_j),$$

the series (5.9) terminates and yields the N-soliton solutions [65], see also [66, Section 1.8] for details, as given in the next proposition.

Proposition 5.6. *Let $0 < k_1 < \ldots < k_N$ and $\varphi_1, \ldots, \varphi_N \in \mathbb{R}$. The N-soliton solution of the bilinear form (5.8) of the Korteweg-de Vries equation is given by*

$$f(x,t) = \sum_{\mu=0,1} \exp\Big(\sum_{j=1}^{N} \mu_j \eta_j + \sum_{i<j}^{(N)} \mu_i \mu_j A_{ij}\Big),$$

where

$$\begin{aligned} \eta_j &= k_j x + k_j^3 t + \varphi_j \\ exp(A_{ij}) &= \left(\frac{k_i - k_j}{k_i + k_j}\right)^2, \end{aligned}$$

and where the sum is taken over all possible vectors $\mu = (\mu_1, \ldots, \mu_N) \in \{0,1\}^N$.

The following proposition shows that the operator method also yields solutions of Hirota's bilinear form.

Proposition 5.7. *Let A, B be $N \times N$-matrices such that $AB + BA$ is one-dimensional, and define $L(x,t) = \exp(Ax + A^3 t)B$. Then*

$$f(x,t) = \det\big(I + L(x,t)\big)$$

is a solution of the bilinear Korteweg-de Vries equation (5.8).

Proof. On $\Omega = \{(x,t) \in \mathbb{R}^2 \mid f(x,t) \neq 0\}$, it has been shown in the proof of Proposition 5.1, that $V = (I+L)^{-1}(AL + LA)$ solves the noncommutative Korteweg-de Vries equation $V_t = V_{xxx} + 3V_x^2$. Using the technique introduced in Section 5.3.2, we conclude that $v = \text{tr}(V)$ solves the scalar potential Korteweg-de Vries equation $v_t = v_{xxx} + 3v_x^2$, and that $v = 2\partial_x \log f$. Together this implies that f satisfies the bilinear form of the Korteweg-de Vries equation (5.8).

Since f is real-analytic, $\mathbb{R}^2 \backslash \Omega$ is a closed set which is nowhere dense in $\mathbb{R}^2$ (or all of $\mathbb{R}^2$). Thus (5.8) holds on all of $\mathbb{R}^2$ for continuity reasons. □

5.6 On traces on operator ideals

In Section 5.3.2, we have seen that the trace plays a key role in constructing solution formulas. In order to generalize the formulas from matrix-parameters to operator-parameters, one needs the concept of traces on quasi-Banach operator ideals [74].

We start from the class $\mathcal{F}$ of finite operators. Let E, F be Banach spaces. An operator $T \in \mathcal{L}(E,F)$ is called *finite*, $T \in \mathcal{F}(E,F)$, if it has finite-dimensional range.

In particular, for $0 \neq a \in E'$, $y \in F$, the operator $a \otimes y : f \mapsto \langle f, a\rangle y$, where $\langle f, a\rangle$ denotes the evaluation of the functional a on f, has one-dimensional range, and $\|a \otimes y\| = \|a\|\|y\|$.

Obviously, linear combinations of one-dimensional operators are finite operators. Conversely, every operator $T \in \mathcal{F}(E,F)$ admits a so-called *finite representation*

$$T = \sum_{j=1}^{n} a_j \otimes y_j$$

with $a_1, \dots, a_n \in E'$, $y_1, \dots, y_n \in F$.

The class of finite operators $\mathcal{F} = \cup_{E,F}\mathcal{F}(E,F)$ has not only the property that each component $\mathcal{F}(E,F)$ is a subspace of $\mathcal{L}(E,F)$, but also the ideal property, namely that multiplying a finite operator with a bounded operator from either side again gives a finite operator. In short, the finite operators form an *operator ideal* in the sense of Pietsch [73], in fact the smallest operator ideal.

The function tr : $\cup_E \mathcal{F}(E,E) \to \mathbb{C}$ assigning to every operator $T \in \mathcal{F}(E,E)$ the complex number

$$\text{tr}(T) = \sum_{j=1}^{n} \langle y_j, a_j \rangle, \tag{5.10}$$

where $\sum_{j=1}^{n} a_j \otimes y_j$ is an arbitrary finite representation of T, is well-defined [74, Lemma 4.2.2]. Moreover, it is linear on $\mathcal{F}(E,E)$ and satisfies the trace property $\text{tr}(TS) = \text{tr}(ST)$ for all $T \in \mathcal{F}(E,F)$, $S \in \mathcal{L}(F,E)$. This shows that tr is a *trace* on the operator ideal $\mathcal{F}$ in the sense of Pietsch [74]. Note that it is even unique.

Alternatively, the trace on $\mathcal{F}$ can be expressed using the fact that it is *spectral*, meaning that the *trace formula*

$$\text{tr}(T) = \sum_{j=1}^{N} \lambda_j(T) \tag{5.11}$$

holds for every T, where $\lambda_j(T)$ denote the non-zero eigenvalues of T counted with multiplicity [74, Theorem 4.2.15].

Both of the two possibilities (5.10), (5.11) to represent the trace on $\mathcal{F}$ indicate a natural approach to extend the trace to larger quasi-Banach operator ideals. In the sequel we hint at some obstacles and results in this direction. Actually the general extension problem is very subtle. For a more detailed report, we refer to [58].

5.6.1 Spectral traces

In order to define the spectral trace on a quasi-Banach operator ideal $\mathcal{A}$, we need to associate to every operator $T \in \mathcal{A}(E, E)$ its *eigenvalue sequence* $(\lambda_j(T))_j$. A rather general class of operators for which this can be done are the so-called *Riesz operators*, see [74, Chapter 3.2] for details. Moreover, we obviously need the eigenvalue sequence to be absolutely summing.

We say that the quasi-Banach operator ideal $\mathcal{A}$ has *eigenvalue type* ℓ_1 if every operator $T \in \mathcal{A}(E, E)$ is a Riesz operator with eigenvalue sequence in ℓ_1. In this situation, a deep result of White [88] states that the trace formula indeed defines a continuous trace on $\mathcal{A}$.

However, a counterexample of Kalton [68] shows that in general this trace is not unique.

5.6.2 Traces for nuclear operators

The formula (5.10) motivates to consider the class $\mathcal{N}_r = \cup_{E,F} \mathcal{N}_r(E, F)$ of *r-nuclear* operators ($0 < r \leq 1$). We say that an operator $T \in \mathcal{L}(E, F)$ belongs to $\mathcal{N}_r(E, F)$ if it admits a representation

$$T = \sum_{j=1}^{\infty} a_j \otimes y_j \quad \text{with} \quad \sum_{j=1}^{\infty} \|a_j\|^r \|y_j\|^r < \infty$$

and $a_1, a_2, \ldots \in E'$, $y_1, y_2, \ldots \in F$. $\mathcal{N}_r$ becomes an r-Banach operator ideal with respect to the r-norm

$$\| \, T \mid \mathcal{N}_r \, \| = \inf \Big(\sum_{j=1}^{\infty} \|a_j\|^r \|y_j\|^r \Big)^{\frac{1}{r}},$$

where the infimum is taken over all possible representations of T. In fact, $\mathcal{N}_r$ is the smallest r-Banach operator ideal.

However, as an example by Enflo [59] shows, on $\mathcal{N} = \mathcal{N}_1$ the assignment

$$\operatorname{tr}_{\mathcal{N}}(T) = \sum_{j=1}^{\infty} \langle y_j, a_j \rangle \tag{5.12}$$

is only independent of the chosen representation $T = \sum_{j=1}^{\infty} a_j \otimes y_j$, if one restricts to Banach spaces with approximation property[1]. If this is the case, $\operatorname{tr}_{\mathcal{N}}$ defines a unique continuous trace [74, Theorem 4.7.2]. Note that this trace is not spectral as an example by Enflo [59] shows.

On the other hand, the assignment (5.12) is well-defined on the smaller r-Banach operator ideals $\mathcal{N}_r$ (over the class of all Banach spaces) if $0 < r \leq \frac{2}{3}$, where it yields a unique trace. In view of the fact that $\mathcal{N}_r$ has eigenvalue type ℓ_1 for $0 < r \leq \frac{2}{3}$, this trace is even spectral.

[1] A Banach space E has the *approximation property* if, for all precompact $K \subset E$ and all $\epsilon > 0$ there is a finite operator T such that $\|x - Tx\| \leq \epsilon$ for all $x \in K$

5.6.3 Determinants and their relationship to traces

Similarly as with traces, there is an axiomatic approach to determinants on quasi-Banach operator ideals [74]. The Trace-determinant theorem clarifies their relationship. It states that there is a one-to-one correspondence between continuous traces and continuous determinants on every quasi-Banach operator ideal.

To conclude this section we state the generalization of Lemma 5.5. Let $\mathcal{A}$ be a quasi-Banach operator ideal admitting a continuous determinant δ, and let $T = T(x)$ be an $\mathcal{A}(E)$-valued function which is differentiable and invertible in some open interval J in $\mathbb{R}$. Then, for all $x \in J$, it holds

$$\Big(\delta\big(I + T(x)\big)\Big)^{-1} \frac{d}{dx}\delta\big(I + T(x)\big) = \tau\Big(\big(I + T(x)\big)^{-1} \frac{d}{dx}T(x)\Big)$$

where τ demotes the continuous trace on $\mathcal{A}$ given by the Gâteaux derivative

$$\tau(S) = \lim_{\xi \to 0} \frac{\delta(I + \xi S) - 1}{\xi}$$

for $S \in \mathcal{A}(E, E)$.

5.7 On elementary operators and Sylvester's equation

To apply the solution formula in Theorem 1, we need that $AB+BA$ is one-dimensional. This leads us to consider Sylvester's equation

$$AX + XB = Y. \tag{5.13}$$

Here $A \in \mathcal{L}(E)$, $B \in \mathcal{L}(F)$ are given bounded linear operators on some Banach spaces E, F. In this context Rosenblum's theorem [76] provides a sufficient condition: (5.13) has a unique solution X for each given right-hand side Y provided that $0 \notin \text{spec}(A) + \text{spec}(B)$.

For a generalization of Theorem 1 to operator-parameters, it is decisive to know whether the solution of (5.13) belongs to a quasi-Banach ideal admitting a continuous determinant. To this end, we slightly change viewpoint.

Let $\mathcal{A}$ be a p-Banach operator ideal, $0 < p \leq 1$, and consider the elementary operator $\Phi_{A,B} : \mathcal{A}(F, E) \longrightarrow \mathcal{A}(F, E)$ defined by

$$\Phi_{A,B}(X) = AX + XB.$$

A result of Aden [47], see also Eschmeier [60] for the case $p = 1$, states that, independently of the underlying p-Banach operator ideal $\mathcal{A}$,

$$\text{spec}(\Phi_{A,B}) = \text{spec}(A) + \text{spec}(B).$$

The crucial consequence for the task at hand is, that, under the assumption $0 \notin \text{spec}(A) + \text{spec}(B)$, the operator equation $AX + XB = C$ is not only always solvable, but for the solution $\Phi_{A,B}^{-1}(C)$ it holds that

$$\Phi_{A,B}^{-1}(C) \in \mathcal{A}(F, E)$$

for any p-Banach operator ideal $\mathcal{A}$.

For hints on alternative approaches to solve Sylvester's equation on quasi-Banach ideals without the condition $0 \notin \text{spec}(A) + \text{spec}(B)$, see [58, Sections 2.4 and 3.2]. For a more detailed discussion and applications, we refer to the overview article [50] and references therein.

5.8 Summary of Paper IV

In Article 4, we study the 2d-Toda lattice

$$\frac{\partial^2}{\partial y \partial x} \log(1 + w_n) = -w_{n+1} + 2w_n - w_{n-1},$$

an integrable differential-difference equation for functions $w_n = w(n, x, y)$ in one discrete variable $n \in \mathbb{Z}$ and two continuous variables $x, y \in \mathbb{R}$.

Our main contribution is a solution formula

$$w_n = \frac{p_{n+1} p_{n-1}}{p_n^2} - 1,$$

where

$$p_n = \delta\big(I + B^n e^{-B^{-1}x - By} D A^n e^{Ax + A^{-1}y} C\big).$$

Here the operator C is assumed to lie in a quasi-Banach ideal admitting a generalized determinant δ and to satisfy a rank one condition, see Theorem 3.7 for details. In contrast, there are no assumptions on D.

Observe that this solution formula depends on two operator parameters A and B. Compared to previous results, we succeed in eliminating the extra assumption that A and B are commuting operators (acting on the same Banach space). Furthermore, we introduce the new free operator parameter D.

A part of the proof of independent interest is the derivation of an operator soliton for a noncommutative version of the 2d-Toda lattice,

$$\frac{\partial}{\partial y}\Big((I + V_n)^{-1} \frac{\partial}{\partial x} V_n\Big) = (I + V_{n-1})^{-1}(I + V_n) - (I + V_n)^{-1}(I + V_{n+1}).$$

Furthermore we will generalize our main result to Hirota's bilinear form of the 2d-Toda lattice and illustrate our work by examples and computer plots.

Author's contributions to this paper: Paper IV is based on an idea of my supervisor. I did a smaller part of the research, as writing the explicit proof of the main theorem and conducting computer experiments.

"The water is an example to us, an example..."
Wilhelm Müller (1794–1827)

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Part II: Solitons

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