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# Numerical Optimization of Pacing Strategy in Cross-country Skiing

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## Abstract

When studying events involving locomotive exercise, such as cross-country skiing, one generally assumes that pacing strategies (i.e. power distributions) have a significant impact on performance. In order to better understand the importance of pacing strategies, a program is developed for numerical simulation and optimization of the pacing strategy in cross-country ski racing. This program computes the optimal pacing strategy for an arbitrary athlete skiing on a delineated course. The locomotion of the skier is described by introducing the equations of motion for cross-country skiing. A transformation of the motion equations is carried out in order to improve the simulation. Furthermore, a nonlinear optimization routine is connected to the simulation program. Simulation and optimization are performed on a fictional male skier. Results show that it is possible to attain an optimal pacing strategy by simulating cross-country skiing while connecting nonlinear optimization routines to the simulation. It is also shown that an optimal pacing strategy is characterized by minor variations in speed. In our opinion, this kind of optimization could serve as essential preparations before important competitions.

### Keywords

*Optimization · Numerical simulation · Cross-country skiing · Pacing strategy · Power distribution*

## 1 Introduction

Cross-country skiing is a winter sport in which the athlete's ability to cover the course distance in the shortest time possible is of decisive importance in its performance. Because the athlete has a limited ability to generate power, the efforts must be distributed in a rational way. Mathematical modeling of a ski race, combined with efficient nonlinear optimization routines, provides a tool for analyzing how these efforts should be best distributed. Carlsson et al. (2011) showed how numerical simulations in cross-country skiing can be performed in order to calculate the total race time for an arbitrary athlete. The simulation model solves the equations of motion for constant time steps. This has the disadvantage of giving us an inexact course length when finishing the race. As the last time step is solved, the simulated athlete has already passed some distance over the finish line. In order to keep this distance as low as possible, very small time steps are required. However, if the equations of motion are expressed as a function of travelled distance, the athlete will always finish at the exact course distance. Additionally, this enables the use of an

adaptive step length estimator which shortens the simulation time dramatically, as well as increase the accuracy.

van Ingen Schenau and Cavanagh (1990) and Moxnes and Hausken (2008) described the equations of motion in endurance sports and cross-country skiing respectively. Carlsson et al. (2011) came up with a model which can simulate a skier along a predetermined course profile by utilizing these motion equations. These equations are generally built up by expressions that involve parameters derived from both internal and external factors. Internal factors adhere to the athlete and are for example energy expenditure, body mass, drag area, etc. External factors are linked to the surrounding environment, such as air density, snow conditions, inclination, wind velocity, etc.

There has been extensive research into pacing strategies (i.e. power distributions) for constant external factors (Foster et al. 1993; Foster et al. 1994; Liedl et al. 1999; Hettinga et al. 2006; Hettinga et al. 2007; Hanon et al. 2008; Lima-Silva et al. 2010; Hettinga et al. 2010; Thomas et al. 2011). Atkinson et al. (2000; 2007b) and Swain (1997) studied how varying external conditions (wind and inclination) influence the optimal choice of pacing strategy in cycling. All suggest that a variable pacing strategy is beneficial if external conditions are changing along the course. Therefore, increased opposing forces from wind and inclination result in a corresponding rise of the current propulsive power. However, none of the mentioned studies state the exact magnitude of propulsive power alteration to make an optimal pacing strategy. Furthermore, all preceding simulations of this type have been performed on made-up course profiles with no connection to real course profiles. de Koning et al. (1999) and Hettinga et al. (2011) optimized the pacing strategy in track cycling (1 km individual time trial and 4 km pursuit) and speed skating (1500 m) respectively, by iterative calculations using a broad range of variable data. As these events usually take place in a controlled indoor environment, no variations in external factors were modeled. They determined the optimal pacing strategy by varying three variables simultaneously. As far as we know, no one has studied optimal pacing strategies in cross-country skiing and no study has investigated the potential of utilizing non-

linear iterative optimization routines to optimize pacing strategies in locomotive sports.

The aim of this study was to design a model which can calculate the optimal pacing strategy for a predetermined athlete on a hilly course in an individual start cross-country skiing sprint competition.

## 2 Method

In order to develop a model of a cross-country skiing course that is suitable for numerical simulations, assumptions are made about a 'straightened' two-dimensional course. The disadvantage, compared to a complete three-dimensional course, is that any effect from turns in the excluded direction is not considered. Moreover, the model does not account for any inertial forces in that third direction, nor the decelerating forces associated with sharp turns. The two-dimensional assumption enables us to express the course as a connected chain of cubical splines. This makes it easy to calculate both the inclination and the curvature at any point along the course. Equations of motion are derived from the forces acting on the skier, as well as the propulsive force generated by the skier. These motion equations are transformed into a system of connected first order differential equations that are implemented into a MATLAB program and solved with the Runge-Kutta-Fehlberg method.

Input data for solving the differential equations are the skier's mass, starting speed  $v$  and available propulsive power  $P$  during the various parts of the race. The propulsive power considered in the model is the external mechanical power generated by the athlete. When there is no acceleration, this corresponds to the speed multiplied by the sum of external forces acting on the athlete. The propulsive power is available from the athlete's rate of energy expenditure  $\dot{E}$ , subsequently reduced by the mechanical efficiency for skiing,  $\eta$ . The mechanical efficiency derives from the external work done divided by the energy expenditure of the athlete. The waste energy here is unspecified but include losses due to friction inside the body (heat) and the movement of limbs (internal work) as well as deformation of the skis, poles and snow (deformation energy) etc. Input data also includes the course profile expressed

as a connected chain of cubical splines, the glide friction coefficient  $\mu$  and drag area  $C_D A$  for standing and semi-squatting postures. For simulated input values, see section 2.6.

## 2.1 Course and athlete

The planned course for the Swedish national sprint championships in 2007 is modeled as a chain of 36 connected cubical splines (Fig. 1). The course length is 1425 m and the total climb is 29.5 m. Skiing is simulated for an imaginary world class male athlete with a body mass of  $m_b = 78$  kg. The athlete's equipment mass is set to  $m_{eq} = 4$  kg in total. The equipment mass includes all the gear that an athlete wears during a cross-country skiing competition. The total mass of athlete and his equipment is  $m_{tot} = 82$  kg.

Simulation is performed on a freestyle sprint qualification race which is an individual start race. This implies that the athlete does not benefit from the reduced drag that can be attained when the skier is situated behind another competitor (Spring et al. 1988).

## 2.2 Forces and scaling

The propulsive force  $F_s$  generated by the athlete is expressed as the propulsive power divided by the current speed in the direction of the course ( $F_s = P/v$ ). This effective propulsive force consists of the effective component of the forces generated through both the skis and poles. Consequently, the effective propulsive force acts in the direction of the course (the s-direction in Fig. 2) and the propulsive power is the product of the speed  $v$  and the propulsive force  $F_s$ .

The external forces acting on the athlete are the gravitational force, the frictional force between skis and snow and the air resistance (i.e. drag). The gravitational force is expressed as  $F_g = m_{tot}g$ , where  $m_{tot}$  is the skiers total mass and  $g$  is the gravity acceleration. The frictional force is expressed as  $F_\mu = \mu N$  where  $N$  is the normal force between the snow and the ski (including centripetal forces) and  $\mu$  is the friction coefficient. If no environmental wind is present, the air resistance is expressed as  $F_D = \frac{1}{2} C_D A \rho v^2$ , where  $C_D$  is the drag coefficient,  $A$  is the projected frontal area,  $\rho$  is

the air density and  $v$  is current speed. The drag force acts in the direction of the course but opposite to the propulsive force. The drag area  $C_D A$  is determined by scaling, using the reference value for an 80 kg athlete which is  $0.65 \text{ m}^2$  in the upright posture and  $0.27 \text{ m}^2$  in the semi-squatting posture (Spring et al. 1988) (Fig. 3).

Scaling is performed using the equation:

$$\frac{C_D A}{C_D A_{ref}} = \left( \frac{m_b}{m_{b_{ref}}} \right)^{2/3} \quad (1)$$

where  $C_D A$  is the requested drag area,  $C_D A_{ref}$  is the reference drag area,  $m_b$  is the skier's body mass and  $m_{b_{ref}}$  is the reference skier's body mass. The drag areas are assumed to scale like the projected frontal areas on their own, which behave like body masses raised to  $2/3$ . This is true for homogenous scaling, however deviations from this scaling is not analyzed in this study.

Certain restrictions are set to decrease the mechanical efficiency of the athlete at high speed. An increased travelling speed requires an equivalent raise in the muscles contraction velocity. Too high contraction velocity will dramatically decrease the muscle force, thus reducing the mechanical efficiency of the athlete. Consequently, the propulsive power is expressed as:

$$P = \eta \varphi \dot{E} \quad (2)$$

where  $\eta$  is the base value of mechanical efficiency,  $\dot{E}$  is the rate of energy expenditure and the reducing function  $\varphi$  (Fig. 4) is calculated as:

$$\varphi = \frac{1}{2} - \frac{1}{\pi} \tan^{-1}(B(v - v_{lim})) \quad (3)$$

where,  $B$  is a parameter that controls the shape of the function,  $v$  is the current speed and  $v_{lim}$  is the limit speed where the efficiency is reduced to the half of  $\eta$ . Varying mechanical efficiency associated with different skiing techniques (classic or freestyle) as well as different gears (e.g. double poling and diagonal stride) (Sidossis et al. 1992; Sandbakk et al. 2010), are however not considered in this study.

A restriction is also constructed for the athlete to crouch when speed rises. It incorporates the above reducing function  $\varphi$ . The drag area ( $C_D A$ ) is reduced when the

athlete crouches to a semi-squatting posture from an upright posture (Spring et al. 1988) (see (4) and Fig. 4). The effective drag area  $C_D A_\varphi$  is calculated as:

$$C_D A_\varphi = C_D A_{URP} \varphi + C_D A_{SSP} (1 - \varphi) \quad (4)$$

where  $C_D A_{URP}$  is the athlete's drag area in the upright posture and  $C_D A_{SSP}$  is the drag area in the semi-squatting posture. The reduced functions in (2) and (4) are synchronized in speed.

### 2.3 Derivation of the equations of motion

Because the inclination of a cross-country skiing course changes continuously, it is useful to describe the equations of motion in the natural directions of movement, normal and tangential to the course, see Figure 2. However, in the simulation it is beneficial to solve the motion equations in the same global directions at every point along the course (Carlsson et al. 2011). Therefore, equations of motion are transformed to the global  $x$ - and  $y$ -coordinates.

Speed and acceleration relationships are shown in (5) and geometric relationships from the course profile equation are shown in (6):

$$\dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2} \quad (5)$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{bmatrix} \cdot \begin{bmatrix} \ddot{s} \\ \ddot{n} \end{bmatrix}$$

$$\alpha = \tan^{-1}(y') \quad (6)$$

$$\frac{1}{R} = \frac{y''}{[1 + (y')^2]^{3/2}}$$

where  $s$  and  $n$  are local coordinates for tangential and normal directions respectively (Fig. 2) and the dots denote differentiation by time. Transformation to the global  $x$ - and  $y$ -coordinates gives the following equations of motion:



$$m_{tot}\ddot{x} = m_{tot}\ddot{s} \cos \alpha + m_{tot}\ddot{n} \sin \alpha = F_S \cos \alpha - F_\mu \cos \alpha - FD \cos \alpha - N \sin \alpha, \quad (7)$$

$$m_{tot}\ddot{y} = m_{tot}\ddot{s} \sin \alpha - m_{tot}\ddot{n} \cos \alpha = F_S \sin \alpha - F_\mu \sin \alpha - FD \sin \alpha + N \cos \alpha - m_{tot}g.$$

where the frictional force is  $F_\mu = \mu N$ , normal force is  $N = m_{tot} \left( g \cos \alpha - \frac{\dot{s}^2}{R} \right)$ , propulsive force is  $F_S = \frac{P}{\dot{s}}$  and the drag force is  $F_D = \frac{1}{2} C_D A \rho \dot{s}^2 = k \dot{s}^2$  where  $k = \frac{1}{2} C_D A \rho$ . For simplicity reasons, no environmental wind is considered in the equations of motion. For consideration of environmental wind, the expression for the drag force will change according to appendix. By inserting the above mentioned force expressions (without environmental wind) into (7) and dividing by the skier's mass  $m_{tot}$  gives the following set of second order differential equations:

$$\ddot{x} = \frac{P}{m_{tot} \dot{s}} \cos \alpha - \frac{k}{m_{tot}} \dot{s}^2 \cos \alpha - \left( g \cos \alpha - \frac{\dot{s}^2}{R} \right) (\mu \cos \alpha + \sin \alpha), \quad (8)$$

$$\ddot{y} = \frac{P}{m_{tot} \dot{s}} \sin \alpha - \frac{k}{m_{tot}} \dot{s}^2 \sin \alpha - \left( g \cos \alpha - \frac{\dot{s}^2}{R} \right) (\mu \sin \alpha + \cos \alpha) - g.$$

## 2.4 Transformation of the equations of motion

From (6),  $\alpha$  and  $R$  depend on  $y'$  and  $y''$ , and from (5),  $\dot{s}$  can be expressed in terms of  $\dot{x}$  and  $\dot{y}$ . So (8) is a system of ordinary differential equations for  $x$  and  $y$  as functions of course time  $t$ . In addition, the course profile specifies  $y$  as a function of  $x$ , which means that the equation for  $\ddot{y}$  is superfluous. Thus the system can be reduced to one second order ordinary differential equation for  $x$  as a function of  $t$ .

The problem with using (8) in a numerical scheme is that the endpoint, the total race time  $T$ , of the interval in  $t$  is not known beforehand. In fact, estimating  $T$  is the main

point of the scheme. If  $t$  is discretized in time intervals of a given length, there will be an overshooting error in  $T$  since the position of the skier is only checked at the end of each time subinterval. Also, the singularity at  $\dot{s} = 0$  in the first term of the right-hand side might need high numerical resolution to avoid instabilities for low speeds.

Fortunately, (8) can be reformulated using  $t$  as a function of  $x$  instead. The inversion fails whenever  $\dot{x} = \frac{dx}{dt} = 0$ , which corresponds to  $\dot{s} = 0$ , where the system in (8) is singular. But the singular term is proportional to  $1/\dot{s}$  and will dominate the other terms for small  $\dot{s}$ . This will increase  $\ddot{x}$  and drive the solution away from  $\dot{s} = 0$ . So  $\dot{x}$  will never reach 0, and the interesting case is of course when  $\dot{x}$  is strictly positive. We can conclude that, with the inverse function theorem, the inverse  $t(x)$  of  $x(t)$  is well defined.

Differential identities are needed to rewrite (8). Differentiating  $x = x(t)$  implicitly with respect to  $x$  gives:

$$1 = \frac{dx}{dt} \frac{dt}{dx} = \dot{x} t' \quad \text{or} \quad \dot{x} = \frac{1}{t'} \quad (9)$$

where the prime denotes differentiation by  $x$ . Differentiating again gives:

$$0 = \ddot{x}(t')^2 + \dot{x}t'' \quad \text{or by using (9) } \ddot{x} = -\frac{1}{(t')^3}t'' \quad (10)$$

Also, from (5) and (6) and using that  $t' = 1/\dot{x} > 0$  and  $\cos \alpha > 0$ ,

$$\dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\sqrt{1+(y')^2}}{t'} = \frac{1}{t' \cos \alpha}. \quad (11)$$

Using (10) and (11) transforms the first part of (8) into:

$$t'' = -\frac{P}{m}(t')^4 \cos^2 \alpha + \frac{k}{m \cos \alpha} t' + t'(g(t')^2 - y'')(\mu + \tan \alpha \cos 2\alpha). \quad (12)$$

Note that the right-hand side of (12) is a fourth order polynomial in  $t'$  without singularities.

The second order differential equation in (12) can be transformed into a system of two first order equations by the standard method of introducing a new variable for  $t'$ .

Subsequently, the system may be solved by a standard numerical solver (Carlsson et al. 2011).

Since the transitions between different splines can now occur at given points in  $x$ , it is possible to use adaptive step refinement to significantly speed up the simulation process. The step length estimator makes the step length longer on parts of the course where the change in inclination is small, and shorter on parts of the course where the inclination changes more rapidly.

## 2.5 Optimization

Optimization is performed on the numerical simulation program using the Method of Moving Asymptotes (MMA) (Svanberg 1987). The optimization routine is set to minimize the total race time  $T$  (Objective function) by varying the propulsive power  $P_j$  at predetermined positions along the course, see (13). In mathematical terms, the optimization problem is formulated as:

Minimize

$$T = \sum_{i=1}^K \Delta t_i \quad (13)$$

subject to the constraints

$$P_{min} \leq P_j \leq P_{max} \quad j = 1, 2, \dots, N \quad (14)$$

$$\frac{1}{T} \int_0^T P(t) dt \leq \bar{P} \quad (15)$$

where  $T$  is the total race time,  $\Delta t_i$  is the time segment during iteration  $i$ ,  $K$  is the total number of time segments during the actual simulation,  $P_j$  is the  $j$ :th optimization variable (i.e. propulsive power at  $j$ ),  $P_{min}$  and  $P_{max}$  are the minimum and maximum available propulsive power for the optimization variables and  $\bar{P}$  is the mean propulsive power limit for the simulated skier. The positions for the optimization variables are selected so that they are closer to each other where the slope is changing rapidly and further apart in areas of more constant inclination. The propulsive power between these positions is available from linear interpolation

between the variables and subsequently reduced by  $\varphi$ . The number of optimization variables is  $N = 75$ .

To prevent the skier from generating unnaturally high propulsive power over a longer period of time, a constraint for mean propulsive power ( $\bar{P}$ ) is added, see (15). Considering that the athlete is moving forward his propulsive power will be equal to, or larger than zero. On the other hand, the athlete can not generate unlimited high propulsive power. Therefore, a global constraint for a minimal and maximal propulsive power is introduced, see (14). In order to stabilize the numerical process, all constraints were normalized during the optimization.

## 2.6 Simulation input data

Simulation and optimization were performed on a fictional 78 kg male skier. The imaginary athlete has a specific  $\dot{V}O_2\text{max}$  of  $76 \text{ ml}\cdot\text{kg}^{-1}\cdot\text{min}^{-1}$ , which is typical for a world class male cross-country sprint skier. This gives him a  $\dot{V}O_2\text{max}$  of  $5.93 \text{ l}\cdot\text{min}^{-1}$  and a maximal rate of aerobic energy expenditure of about 2086 W. Considering a typical male cross-country skier, his mechanical efficiency will be about  $\eta = 0.15$  in the freestyle technique (Ainegren et al. 2012) resulting in a maximal aerobic propulsive power of 313 W. In the current study a mean propulsive power of 120% of the maximal aerobic propulsive power is used. Brickley et al. (2007) reported a similar value of 107% for trained cyclists performing an all-out exercise for about 3 min. The higher percentage makes sense considering that an elite cross-country sprint skier is considered to have a relatively greater anaerobic capacity than the less anaerobically trained cyclist. This together with the reducing function  $\varphi$  gives a mean propulsive power of  $\bar{P} = 376 \cdot \varphi \text{ W}$  for (15). The maximum attainable peak propulsive power in (14) is set to 2.5 times the aerobic power at  $P_{max} = 782 \cdot \varphi \text{ W}$  (Brickley et al. 2007).

The two drag areas in (4) for upright posture (URP) and semi-squatting posture (SSP) are calculated to  $C_D A_{URP78} = 0.639 \text{ m}^2$  and  $C_D A_{SSP78} = 0.265 \text{ m}^2$  respectively, using the scaling law in (1). A dynamic friction coefficient of  $\mu = 0.03$  is used in the simulation. It represents a typical tribological interface between polyethylene and snow in temperatures around  $-1^\circ\text{C}$  to  $-5^\circ\text{C}$  in dry snow (Buhl et al.

2001). No wind is assumed. The air density is estimated at  $\rho = 1.3163 \text{ kg}\cdot(\text{m}^3)^{-1}$  considering skiing at sea level (the course is located near the town of Hudiksvall on the east coast of Sweden) at  $-5 \text{ }^\circ\text{C}$  with air pressure at 101.325 kPa. The starting speed is set to  $3 \text{ m}\cdot\text{s}^{-1}$  in the course direction (the s-direction in Fig. 2). For the reducing function, the shape parameter is set to  $B = 4$  and the speed limit is set to  $v_{lim} = 10 \text{ m}\cdot\text{s}^{-1}$  ( $36 \text{ km}\cdot\text{h}^{-1}$ ) derived from Andersson et al. (2010).

### 3 Results

Optimization starts with initial values of  $P_j = \bar{P} = 376 \cdot \varphi \text{ W}$  for all variables  $P_j$ . After 20 iterations, an optimized propulsive power distribution has been calculated that gives the total time of 3 min 7.4 s. This must be compared to the time for an more or less even propulsive power distribution, where every variable is set to  $P_j = \bar{P} = 376 \cdot \varphi \text{ W}$ . This pacing strategy gives a total race time of 3 min 20.4 s. Consequently, the optimized pacing strategy gives a time gain for this athlete of 13.0 s. All constraints are still fulfilled and the divergence from the stipulated reference value of the mean propulsive power in (15) is less than 0.0021 W. The optimized pacing strategy for a 78 kg male athlete is presented in Figure 5.

The propulsive mean power ( $\bar{P}$ ) that the athlete is able to sustain for a longer period of time is shown as a dotted line in Figure 5. Because of the formulation of the constraint in (15), the area within the line for the propulsive power distribution should be equal above and below the mean propulsive power (provided that this constraint is fulfilled with equality). The athlete's average speed in the simulation is  $7.60 \text{ m}\cdot\text{s}^{-1}$  ( $27.38 \text{ km}\cdot\text{h}^{-1}$ ). Minimum speed is equal to the starting speed  $3 \text{ m}\cdot\text{s}^{-1}$  ( $10.80 \text{ km}\cdot\text{h}^{-1}$ ) and maximum speed is  $11.75 \text{ m}\cdot\text{s}^{-1}$  ( $42.32 \text{ km}\cdot\text{h}^{-1}$ ).

### 4 Discussion

The results of this study confirm the results of Swain (1997), i.e. that major time savings can be realized by utilizing a variable pacing strategy. The propulsive power distribution in Figure 5 clearly indicates that the optimization routine strives to increase propulsive power in the uphill slopes and decrease it in the downhill slopes. Palmer et al. (1999) reported a significant increase in blood lactate concentration

when altering exercise intensity (40-80%  $\dot{V}O_2\text{max}$ ) compared to a similar constant intensity cycling time trial. Contradictory results have been presented by Atkinson et al. (2007a) and Liedl et al. (1999), showing no significant alteration in either heart rate,  $\dot{V}O_2$ , blood lactate, perceived exertion or pedal rate when altering the propulsive power  $\pm 5\%$  from the propulsive mean power in time trial cycling (1-hour and 800-kJ). This propulsive power interval is considerably narrower than the whole range from 0 W to  $P_{max}$  applied in this study. With that in mind the result reported from Palmer et al. (1999) would be more applicable to this kind of simulation.

Consequently, the constraint for mean propulsive power limit in (15) may be reformulated to consider the increased accumulation of fatigue-related substances in variable pacing strategies.

Figure 5 clearly shows that the athlete benefits from maintaining his speed as constant as possible in order to optimize his pacing strategy. This is easy to realize when considering that great variations in speed yield a higher average drag force than even pacing. This is due to the exponential behavior of the expression for the drag force  $F_D$ .

It might seem strange that an optimized pacing strategy lowers the propulsive power at the very end of the race. In real cross-country skiing it is common sense to keep the propulsive power high in the end of a race, so as to finish as fast as possible. The kinetic energy at the end of the race will be wasted after the finish line. Therefore, it is rational to use higher propulsive power at earlier stages of the race. This implies that the optimization routine, due to the constraint in (15), iterates to a lower propulsive power at the very end of the race while increasing the propulsive power along sections of greater impact. Obviously, the inclination in the last sections of the course also affects this energy-saving phenomenon.

The assumption that mechanical efficiency decreases rapidly at a  $v_{lim}$  of 10 m/s can be supported by d-GNSS data presented by Andersson et al. (2010), which shows at which speed every “gear” (Nilsson et al. 2004) is performed. Judging by the deviation of gear speed in Andersson et al. (2010),  $v_{lim}$  in (3) might be treated as an individual parameter, one that is probably affected by the size of the athlete as well

as his rapidness and technical skills. Skiers with longer limbs are more likely to be more effective at high speed due to a reduced cycle-rate (frequency) compared to skiers with shorter limbs.

The time saved from utilizing an optimal pacing strategy, as in this study, compared to an even pacing strategy, is 13.0 s which is a reduction of 6.5%. In the men's individual sprint qualification in the 2010 Olympics, this would be the difference between placing number 51 and winning the qualification. However, one can expect that today's skiers have already opted for an improved pacing strategy compared to an even propulsive power distribution, but presumably not an optimal one.

The simulated athlete's average speed of  $7.60 \text{ m}\cdot\text{s}^{-1}$  may be compared to the results in the men's individual sprint qualification (classic technique) in the 2010 Olympics in Vancouver, Canada. The winner of the qualification had an average speed of  $6.85 \text{ m}\cdot\text{s}^{-1}$  over the 1470 m course. Of course those races are not equivalent when it comes to skiing technique, course profile and environmental conditions, but the athletes' body masses are equal in those two cases. The total climb on the courses is 44 m and 29.5 m for the Olympic course and the simulated course respectively. No measurements of gliding friction were made on the race day of the individual sprint competitions in the 2010 Olympics but the glide friction in the simulation ( $\mu = 0.03$ ) is considered to be typical for snow temperatures around  $-1^\circ\text{C}$  to  $-5^\circ\text{C}$ . This, in combination with the lower total climb and a different skiing technique (skating vs. classic), should contribute to the somewhat higher average speed in the simulation compared to the Olympic example. However, in the study of Andersson et al. (2010), a real cross-country sprint race (skating technique) was performed with the fastest athlete skiing an average speed of  $7.39 \text{ m}\cdot\text{s}^{-1}$ . That is 2.8% lower than the simulated athlete in the present study. The course in the study of Andersson et al. (2010) had the same length (1425 m) as the one simulated in this study but had a greater total climb of 52 m. The difference in total climb may be one explanation of the lower average speed. Furthermore, that study used the same skiing technique (skating technique) and had similar environmental conditions ( $-2^\circ\text{C}$ , no wind) as the present study. This does not clarify the significance of the numerical model, but it shows that the results are reasonable.

In order to analyze the robustness of the optimization model, calculations are performed with different starting values, such as: glide friction, body mass, mean and maximum propulsive power (not presented in this study). However, this results only in small variations of the end time and distribution of power.

The skier's ability to perform can of course vary from day to day and this "on the day" performance can have a great effect on the results. The simulations shown here do not take such variations into account, but it is still of great interest to study how the propulsive power should be distributed in the most efficient way during a cross-country skiing race. Once the profile of a course is known, such studies might serve as an essential part of preparations for important championships. Athletes with known available propulsive power, drag area, mass etc. can be simulated to determine their optimal pacing strategy. However, the question is how to convey these results to an actual athlete. As there is no direct measurement of the propulsive power while skiing, it is hard if not impossible to follow the optimized distribution of power. Therefore, a more practical approach is to follow the optimized speed distribution, as this can be measured more easily. Subsequently, this can be transformed to visual or audio based stimuli in training.

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## Appendix

To model environmental wind, the drag force would be expressed as  $F_D =$

$$\frac{1}{2} C_D A (\dot{s}, w, \beta) \rho \sqrt{(\dot{s} + w \cos \beta)^2 + (w \sin \beta)^2} (\dot{s} + w \cos \beta),$$

where  $\dot{s}$  is the speed in the direction of the course,  $w$  is the wind speed,  $\beta$  is the angle between the direction

of travel and the environmental wind direction and  $\rho$  is the air density.  $C_D A$  is the drag area and it depends on the quantities of  $\dot{s}$ ,  $w$  and  $\beta$ .

### Figure captions

**Fig. 1** Simulated course profile from the planned Swedish national sprint championships 2007

**Fig. 2** Arbitrary section of the course with forces on the skier and chosen coordinate system

**Fig. 3** Schematic view of the upright posture (left) and semi-squatting posture (right)

**Fig. 4**  $\eta \cdot \varphi$  and  $C_D A_\varphi$  as functions of speed for a 78 kg male skier with  $B = 4$  and  $v_{lim} = 10 \text{ m}\cdot\text{s}^{-1}$

**Fig. 5** Course profile, speed, optimized propulsive power distribution and mean propulsive power for a world class male skier of 78 kg