

Intercalates in double and triple arrays

Tomas Nilson 

Department of Mathematics and Science Education, Mid Sweden University, Sundsvall, Sweden

Correspondence

Tomas Nilson, Department of Mathematics and Science Education, Mid Sweden University, Mittuniversitetet, Sundsvall 851 70, Sweden.
Email: tomas.nilson@miun.se

Abstract

This paper addresses the question of how intercalates occur in the two known infinite families of triple arrays, the Paley triple arrays constructed in 2005 by Preece et al., and the Triple arrays from difference sets in 2017 by Nilson and Cameron. The main reason for doing this is that the number of such embedded Latin squares is often used when checking whether two arrays are isotopic or not. We determine sharp bounds for the number and density of intercalates for the main subclasses of these families respectively. We also prove the existence of an infinite family of triple arrays in which every two occurrences of an entry lie in an intercalate.

KEYWORDS

difference set, double array, intercalate, symmetric block design, triple array

1 | INTRODUCTION

In the 1960s Agrawal [1] amongst others, started to study the type of row-column designs that we now call triple arrays. Agrawal observed that a triple array in the canonical case implies the existence of a symmetric balanced incomplete block design (SBIBD) and suggested a construction method based on the latter. Many examples have been constructed by this method, but the existence question is still open as it has not yet been proven that the construction can always be carried out.

Definition 1.1. A *triple array* is an $r \times c$ array on ω symbols arranged so that no symbol occurs more than once in any row or column, and satisfies the following four conditions

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- TA1. each symbol occurs κ times, $\kappa < r < c$;
- TA2. any two distinct rows contain λ_{rr} common symbols;
- TA3. any two distinct columns contain λ_{cc} common symbols;
- TA4. any row and column contain λ_{rc} common symbols.

For a triple array with the above parameters we use the notation $TA(\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$. An array as above that satisfies conditions TA1–TA3 is called a *double array* for which we use the notation $DA(\omega, \kappa, \lambda_{rr}, \lambda_{cc} : r \times c)$. A double array that does not satisfy TA4 is called a *proper double array*.

A natural question to ask is if two arrays have the same structure, if they are isotopic.

Definition 1.2. Two arrays on the same set of symbols are *isotopic* to one another if one can be obtained from the other by some combination of the operations: permutation of rows, permutations of columns and permutation of symbols. Two isotopic arrays belong to the same *isotopy class*.

When examining whether two arrays are isotopic, or determining the number of isotopy classes, it is common to compare the number of intercalates.

Definition 1.3. A 2×2 Latin square embedded in a larger array is an *intercalate*. The two rows of an intercalate need not be adjacent, nor need the two columns (Figure 1).

Intercalates are considered because the number of such is preserved under permutation of rows, columns and symbols. For an array A , we denote the number of intercalates in A by $N(A)$.

In this paper, we investigate the occurrences of intercalates in the two known infinite families of triple arrays. In Section 2 we give preliminary results for triple arrays. In Section 3 we make some general observations about intercalates and consider small arrays. In Section 4 we recall the construction of Paley triple arrays and investigate how intercalates can occur in this family and how they are distributed. We also determine bounds for the number of intercalates and maximal density. In Section 5 we do the corresponding investigation for intercalates in the family of triple arrays constructed from difference sets in finite groups. Furthermore, we prove the existence of an infinite family of triple arrays in which every two occurrences of a symbol lie in an intercalate. In this section, it is also natural to include double arrays. Finally, in Section 6 we give some concluding remarks.

2 | PRELIMINARIES

We recall some facts about double and triple arrays, several of which were proved by McSorley et al. [6].

1	2	3	4	5	6	7	8	9	10
2	1	11	12	14	15	9	10	7	8
3	11	1	13	9	10	14	15	5	6
4	12	13	1	8	7	6	5	15	14
15	6	8	9	13	2	12	3	4	11
14	5	7	10	2	13	3	12	11	4

FIGURE 1 A $TA(15, 4, 6, 2, 4 : 6 \times 10)$ with 45 intercalates. It is exceptional as every two occurrences of an entry lie in an intercalate

Theorem 2.1 (McSorley et al. [6]). *Any DA($\omega, \kappa, \lambda_{rr}, \lambda_{cc} : r \times c$) satisfies*

- (1) $\omega\kappa = rc$,
- (2) $\lambda_{rr}(r - 1) = c(\kappa - 1)$,
- (3) $\lambda_{cc}(c - 1) = r(\kappa - 1)$,
- (4) $\lambda_{rr}r(r - 1) = \lambda_{cc}c(c - 1)$.

Theorem 2.2 (McSorley et al. [6]). *Any triple array TA($\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$) with $\kappa \neq r$ and $\kappa \neq c$ satisfies $\omega \geq r + c - 1$.*

Here we consider the extremal case where $\omega = r + c - 1$ of double and triple arrays for which there are many examples. In the nonextremal case where $\omega > r + c - 1$ there are some proper double arrays known, but only one triple array, a TA(35, 3, 5, 1, 3 : 7 × 15) [6]. The statement that an extremal triple array can always be constructed from a nontrivial SBIBD, is known as *Agrawal's conjecture*.

Conjecture 2.3 (Agrawal [1]). *If there is a ($\omega + 1, r, \lambda_{cc}$)-SBIBD with $r - \lambda_{cc} > 2$, then there is a TA($\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$) with $\omega = r + c - 1$.*

The condition $r - \lambda_{cc} > 2$ excludes the smallest nontrivial SBIBD with parameters (7, 3, 1) and its complement. The converse of the conjecture is true.

Theorem 2.4 (McSorley et al. [6]). *If there is a TA($\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$) with $\omega = r + c - 1$, then there is a ($\omega + 1, r, \lambda_{cc}$)-SBIBD.*

Extremal triple arrays can also be regarded as balanced grids, first defined in [6]. This alternative definition gives us more properties with which to work.

Definition 2.5. Let A be an $r \times c$ array on ω symbols in which every symbol occurs κ times and no symbol occurs more than once in any row or column. We define μ_{xy} to be the number of times that distinct symbols x and y occur in the same row or column of A . If there is a constant μ such that $\mu_{xy} = \mu$ for every x and y then A is a *balanced grid*. We denote such a balanced grid by $BG(\omega, \kappa, \mu : r \times c)$.

McSorley [7] proved the following about balanced grids and triple arrays.

Lemma 2.6 (McSorley [7]).

- (1) *In an extremal triple array TA($\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$) the following are equivalent: $\omega = r + c - 1$ and $\lambda_{rr} = c - \kappa$ and $\lambda_{cc} = r - \kappa$.*
- (2) *In a balanced grid BG($\omega, \kappa, \mu : r \times c$) we have $\omega = r + c - 1$ if and only if $\mu = \kappa$.*

Theorem 2.7 (McSorley [7]). *Let $\omega = r + c - 1$. Then every triple array is a TA($\omega, \kappa, c - \kappa, r - \kappa, \kappa : r \times c$) and every balanced grid is a BG($\omega, \kappa, \kappa : r \times c$) and they are equivalent.*

We will make use of the following observation.

Remark 2.8. Note that (1) in Lemma 2.6 also holds for extremal double arrays. This is because the proof in [7] only uses Theorem 2.1 and $\omega = r + c - 1$.

3 | SMALL ARRAYS AND GENERAL OBSERVATIONS

In [6] there are lists of all parameter sets for extremal double and triple arrays by size of rows with $2 \leq r \leq 16$, together with the few open cases in this interval. Here we check which arrays are the smallest with intercalates, prove some necessary conditions and define the density of intercalates. We start with double arrays (Figure 2).

Lemma 3.1. *Let A be a $DA(\omega, \kappa, \lambda_{rr}, \lambda_{cc} : r \times c)$ with an intercalate, then $\lambda_{cc} \geq 2$ and $\lambda_{rr} \geq 3$.*

Proof. If there is an intercalate, then $\lambda_{cc} \geq 2$. As $r < c$ identity (4) of Theorem 2.1 gives $\lambda_{rr} \geq \frac{2c(c-1)}{r(r-1)} > 2$ and by the example in Figure 2 we know that the bounds are sharp. \square

Since Lemma 3.1 excludes both $DA(6, 2, 2, 1 : 3 \times 4)$ and $DA(12, 3, 6, 1, 3 : 4 \times 9)$ from having intercalates we know that $DA(10, 3, 3, 2 : 5 \times 6)$ is the smallest. However, for extremal triple arrays we must move up in size because of the following result.

Lemma 3.2. *Let A be an extremal $TA(\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with an intercalate, then $\kappa \geq 4$.*

Proof. If A has an intercalate with symbols x and y , then they meet in two rows and in two columns, so $\mu_{xy} \geq 4$. From Theorem 2.7 we know that A is a balanced grid with $\kappa = \mu$ and the result follows. \square

Hence, the smallest extremal triple array with intercalates is $TA(15, 4, 6, 2, 4 : 6 \times 10)$, and we have already seen one example in Figure 1.

Lemma 3.3. *Let A be a $TA(\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ in which every two occurrences of an entry lie in an intercalate, then $N(A) = \frac{\kappa(\kappa-1)\omega}{4}$.*

Proof. Each of the ω symbols is in $\binom{\kappa}{2}$ intercalates and each intercalate contains two symbols, thus $N(A) = \binom{\kappa}{2} \frac{\omega}{2} = \frac{\kappa(\kappa-1)\omega}{4}$. \square

1	4	7	6	8	10
2	5	6	7	9	1
3	6	5	8	4	9
4	1	2	3	10	5
7	8	9	10	2	3

FIGURE 2 A proper $DA(10, 3, 3, 2 : 5 \times 6)$ with four intercalates

Note that the converse of Lemma 3.3 is also true. We want to compare the density of intercalates in double and triple arrays.

Definition 3.4. Let A be a double array, then the density $\rho(A)$ of intercalates in A is defined by $\rho(A) = \frac{4N(A)}{\kappa(\kappa - 1)\omega}$.

From Figure 1 we know there is a triple array A with full density $\rho(A) = 1$, but no such proper double array exists.

Proposition 3.5. Let A be an extremal proper double array, then $\rho(A) < 1$.

Proof. Suppose A is an extremal proper double array with $\rho(A) = 1$. Then $A(i, j)$ lies in $\kappa - 1$ intercalates which together defines κ common symbols for row i and column j . Note that row i and column j cannot have other common symbols since it would mean that $A(i, j)$ lies in more than the possible number $\kappa - 1$ of intercalates. Hence, each pair of row-column has κ common symbols, so A is a triple array. \square

4 | PALEY TRIPLE ARRAYS

The family of Paley triple arrays was the first infinite family of triple arrays to be constructed. This was done to different degrees by Seberry [13], Street [14], Bagchi [2] and to the full extent by Preece et al. [12] who also named the family.

4.1 | Construction of Paley triple arrays

We first recall the construction given by Preece et al. [12] on which our investigation and results on Paley triple arrays are entirely based.

Let q be an odd prime power and let θ be a primitive element of $GF(q)$. We let Q denote the set of even powers of θ and R the set of odd powers. Further, let $Q_0 = Q \cup \{0\}$ and $R_0 = R \cup \{0\}$.

Construction 4.1 (Preece et al. [12]). Order the elements of $GF(q)$, say by $\{0 = w_0, w_1, \dots, w_{q-1}\}$, and write $GF(q)' = \{0' = w'_0, w'_1, \dots, w'_{q-1}\}$, a duplicate copy. For non-zero elements a and b define the $q \times q$ matrix A_0 by:

$$A_0(i, j) = \begin{cases} w_i - \frac{w_i - w_j}{a} & \text{if } w_i - w_j \in Q, \\ \left(w_i + \frac{w_i - w_j}{b}\right)' & \text{if } w_i - w_j \in R_0. \end{cases}$$

Let A be the $q \times (q + 1)$ matrix obtained by appending $(w_0, w_1, \dots, w_{q-1})$ to A_0 as column q , that is, $A(i, q) = w_i$, for $i = 0, 1, \dots, q - 1$.

If an array given by Construction 4.1 is a triple array, then it is called a *Paley triple array* (PTA) and has parameters $TA\left(2q, \frac{q+1}{2}, \frac{q+1}{2}, \frac{q-1}{2}, \frac{q+1}{2} : q \times (q + 1)\right)$.

Theorem 4.2 (Preece et al. [12]). *Suppose $q \equiv 1 \pmod{4}$. Choose a and b such that $ab \in Q, (a - 1) \in Q$ and $(b + 1) \in R$. Then A is a Paley triple array.*

Theorem 4.3 (Preece et al. [12]). *Suppose $q \equiv 3 \pmod{4}$. Choose a and b such that $(a - 1)(b + 1) \in Q$ and if $(a - 1) \in R$ then $ab \in Q$. The A is a Paley triple array.*

We include two examples of Paley triple arrays that are interesting when we consider intercalates (Figures 3 and 4).

4.2 | Results for Paley triple arrays

The smallest Paley triple array is a $TA(10, 3, 3, 2, 3 : 5 \times 6)$ and by Lemma 3.2 we know that no such triple array has intercalates, regardless of construction method. In the following, we investigate the presence of intercalates in larger Paley triple arrays obtained from the construction in [12].

Lemma 4.4. *Let A be a $q \times (q + 1)$ Paley triple array as specified in Construction 4.1, then A consists of a $q \times q$ array A_0 together with the column q . Suppose A has intercalates, then the following holds.*

- (1) *If $q \equiv 1 \pmod{4}$, then all intercalates have symbols in column q and all symbols in intercalates are of type w .*
- (2) *If $q \equiv 3 \pmod{4}$, then all intercalates are contained in A_0 and each intercalate has one symbol of type w and one of type w' .*

Proof. First, we use Construction 4.1 to see what types of symbols can be in an intercalate contained in A_0 , there are three cases to check,

- (1) both symbols are of type w ,
- (2) both symbols are of type w' ,
- (3) one symbol is of type w and one is of type w' .

Case (1): If there is an intercalate in A_0 and both symbols are of type w , then there are distinct rows i_1, i_2 and columns j_1, j_2 such that

$0'$	$\alpha^{7'}$	α^6	α'	1	$\alpha^{3'}$	α^2	$\alpha^{5'}$	α^4	0
$\alpha^{4'}$	α'	$0'$	1	$\alpha^{5'}$	$\alpha^{2'}$	α^7	α^6	α^3	α
α^6	$1'$	$\alpha^{2'}$	α^5	$\alpha^{7'}$	α^3	0	α'	$\alpha^{4'}$	α^2
$\alpha^{6'}$	1	α^5	$\alpha^{3'}$	$0'$	α^2	$\alpha^{7'}$	$\alpha^{4'}$	α	α^3
1	$\alpha^{3'}$	$\alpha^{6'}$	$\alpha^{2'}$	$\alpha^{4'}$	α^7	α'	α^5	0	α^4
$1'$	$\alpha^{6'}$	α^3	α^2	α^7	$\alpha^{5'}$	$0'$	α^4	α'	α^5
α^2	α^7	0	$\alpha^{5'}$	$1'$	$\alpha^{4'}$	$\alpha^{6'}$	α	$\alpha^{3'}$	α^6
$\alpha^{2'}$	α^6	$\alpha^{3'}$	$1'$	α^5	α^4	α	$\alpha^{7'}$	$0'$	α^7
α^4	α^3	$\alpha^{5'}$	α	0	$\alpha^{7'}$	$\alpha^{2'}$	$\alpha^{6'}$	$1'$	1

FIGURE 3 A $TA(18, 5, 5, 4, 5 : 9 \times 10)$, given as Example 8 in [12]. It has 18 intercalates, all including cells in the last column

0'	3'	6'	6	5'	3	5	0
6	1'	4'	0'	0	6'	4	1
5	0	2'	5'	1'	1	0'	2
1'	6	1	3'	6'	2'	2	3
3	2'	0	2	4'	0'	3'	4
4'	4	3'	1	3	5'	1'	5
2'	5'	5	4'	2	4	6'	6

FIGURE 4 A $TA(14, 4, 4, 3, 4 : 7 \times 8)$, given as Example 11 in [12]. It has 21 intercalates, all within the first 7 columns. Each intercalate contains one primed symbol and one non-primed

$$\begin{cases} w_{i_1} - \frac{w_{i_1} - w_{j_1}}{a} = w_{i_2} - \frac{w_{i_2} - w_{j_2}}{a} \\ w_{i_2} - \frac{w_{i_2} - w_{j_1}}{a} = w_{i_1} - \frac{w_{i_1} - w_{j_2}}{a} \end{cases}$$

giving

$$\begin{cases} a(w_{i_1} - w_{i_2}) = w_{i_1} - w_{j_1} - w_{i_2} + w_{j_2} \\ a(w_{i_1} - w_{i_2}) = w_{i_1} - w_{j_2} - w_{i_2} + w_{j_1} \end{cases}$$

so

$$w_{i_1} - w_{j_1} - w_{i_2} + w_{j_2} = w_{i_1} - w_{j_2} - w_{i_2} + w_{j_1}$$

which gives $w_{j_1} = w_{j_2}$. We have a contradiction since w_{j_1} and w_{j_2} are distinct elements.

Case (2): If there is an intercalate in A_0 and both symbols are of type w' , then

$$\begin{cases} w_{i_1} + \frac{w_{i_1} - w_{j_1}}{b} = w_{i_2} + \frac{w_{i_2} - w_{j_2}}{b} \\ w_{i_1} + \frac{w_{i_1} - w_{j_2}}{b} = w_{i_2} + \frac{w_{i_2} - w_{j_1}}{b} \\ b(w_{i_1} - w_{i_2}) = w_{i_2} - w_{j_2} - w_{i_1} + w_{j_1} \\ b(w_{i_1} - w_{i_2}) = w_{i_2} - w_{j_1} - w_{i_1} + w_{j_2} \end{cases}$$

so

$$w_{i_2} - w_{j_2} - w_{i_1} + w_{j_1} = w_{i_2} - w_{j_1} - w_{i_1} + w_{j_2}$$

which gives $w_{j_1} = w_{j_2}$. We have a contradiction since w_{j_1} and w_{j_2} are distinct elements. Thus, from case (1) and (2) we know that there are no intercalates within A_0 in which both symbols are of the same type.

Case (3): If there is an intercalate in A_0 in this case, then

$$\begin{cases} w_{i_1} - \frac{w_{i_1} - w_{j_1}}{a} = w_{i_2} - \frac{w_{i_2} - w_{j_2}}{a} \\ w_{i_1} + \frac{w_{i_1} - w_{j_2}}{b} = w_{i_2} + \frac{w_{i_2} - w_{j_1}}{b} \end{cases}$$

giving

$$\begin{cases} a(w_{i_1} - w_{i_2}) = w_{i_1} - w_{j_1} - w_{i_2} + w_{j_2} \\ b(w_{i_1} - w_{i_2}) = w_{i_2} - w_{j_1} - w_{i_1} + w_{j_2} \\ (a - 1)(w_{i_1} - w_{i_2}) = w_{j_2} - w_{j_1} \\ (b + 1)(w_{i_1} - w_{i_2}) = w_{j_2} - w_{j_1} \end{cases}$$

so

$$a - 1 = b + 1.$$

If $q \equiv 1 \pmod{4}$ then Theorem 4.2 requires that $(a - 1) \in Q$ and $(b + 1) \in R$ for a triple array, so we have a contradiction and there is no intercalate in A_0 . However, if $q \equiv 3 \pmod{4}$ then it can be possible to choose a and b such that $a - 1 = b + 1$ which was done for the array in Figure 4. \square

It remains to check when an intercalate can have cells in column q . According to Construction 4.1 the symbols of column q are ordered $\{0 = w_0, w_1, \dots, w_{q-1}\}$. Also, an intercalate including cells in column q consist of symbols of type w . Therefore we have such an intercalate if

$$\begin{cases} w_{i_1} - \frac{w_{i_1} - w_j}{a} = w_{i_2} \\ w_{i_2} - \frac{w_{i_2} - w_j}{a} = w_{i_1} \end{cases}$$

giving

$$\begin{cases} a(w_{i_1} - w_{i_2}) = w_{i_1} - w_j \\ a(w_{i_1} - w_{i_2}) = w_j - w_{i_2} \end{cases}$$

so

$$w_{i_1} - w_j = -(w_{i_2} - w_j).$$

Both $w_{i_1} - w_j$ and $w_{i_2} - w_j$ are in Q by construction, so $-(w_{i_2} - w_j)$ needs to be in Q which requires that also $(-1) \in Q$, and by the well known Lemma 4.5 below we know when that is the case.

Lemma 4.5. *In $GF(p^n)$, if $p^n \equiv 1 \pmod{4}$ then $(-1) \in Q$, and if $p^n \equiv 3 \pmod{4}$ then $(-1) \in R$.*

Hence, if $q \equiv 1 \pmod{4}$ there can be intercalates with cells in column q as we have seen in Figure 3. But, if $q \equiv 3 \pmod{4}$ there are no intercalates with cells in column q .

Now we can derive bounds on the number of intercalates in Paley triple arrays. Note that in this family we have $q = r = \frac{w}{2}$.

Proposition 4.6. *Let A be a PTA($\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$) with $r \equiv 1 \pmod{4}$, then $N(A) \leq \frac{(\kappa-1)\omega}{4}$ and the bound is sharp.*

Proof. By Lemma 4.4 we know that all intercalates have symbols in column q , all of their symbols are of type w and there are $\frac{\omega}{2}$ of them. Take one such w in column q , then there are $\kappa - 1$ copies in A_0 with which it can form intercalates. Since each intercalate contains two symbols of type w , the maximal number of intercalates is $\frac{\kappa-1}{2} \cdot \frac{\omega}{2} = \frac{(\kappa-1)\omega}{4}$ and by Figure 3 we know that the bound is sharp. □

Proposition 4.7. *Let A be a PTA($\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$) with $r \equiv 3 \pmod{4}$, then $N(A) \leq \frac{(\kappa-1)(\kappa-2)\omega}{4}$ and the bound is sharp.*

Proof. By Lemma 4.4 we know that all intercalates are contained within A_0 and that each intercalate has one symbol of type w and one of type w' . Since each symbol of type w also occurs once in column q there remain $\kappa - 1$ of each w and there are $\frac{\omega}{2}$ of them. Hence, the maximal number of intercalates is $\binom{\kappa-1}{2} \cdot \frac{\omega}{2} = \frac{(\kappa-1)(\kappa-2)\omega}{4}$ and by Figure 4 we know that the bound is sharp. □

We make some observations about density of intercalates in the two main classes of Paley triple arrays.

Corollary 4.8. *Let A be a PTA($\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c$),*

- (1) *if $r \equiv 1 \pmod{4}$, then $\rho(A) \leq \frac{1}{\kappa}$,*
- (2) *if $r \equiv 3 \pmod{4}$, then $\rho(A) \leq \frac{\kappa-2}{\kappa}$.*

Proof. We take $N(A)$ from Propositions 4.6 and 4.7, respectively. If $r \equiv 1 \pmod{4}$ we get

$$\rho(A) \leq \frac{(\kappa-1)\omega}{4} \cdot \frac{4}{\kappa(\kappa-1)\omega} = \frac{1}{\kappa},$$

and if $r \equiv 3 \pmod{4}$ we get

$$\rho(A) \leq \frac{(\kappa-1)(\kappa-2)\omega}{4} \cdot \frac{4}{\kappa(\kappa-1)\omega} = \frac{\kappa-2}{\kappa}. \quad \square$$

We note that if $q = r \equiv 1 \pmod{4}$, then ρ has a maximal value $\frac{1}{5}$ reached by the array in Figure 3. If $q = r \equiv 3 \pmod{4}$, then the potential maximal values for ρ starts at $\frac{1}{2}$ as in Figure 4 and then increases with the size of the array.

5 | DOUBLE AND TRIPLE ARRAYS FROM DIFFERENCE SETS

The existence of an infinite family of triple arrays constructed from difference sets in finite groups was proven by Nilson and Cameron [11]. The construction is given in two equivalent versions. In the first, a Youden rectangle is developed from a difference set, and then it is used to construct a double or a triple array. Here we will use the second version, the so-called direct construction.

A group can have several nontrivial difference sets, all giving double arrays and some that give triple arrays. We will see that it is group properties that determine the number of intercalates. Therefore, we here give results for both types of arrays, although we are mainly interested in triple arrays.

5.1 | Construction and preliminaries

Definition 5.1. Let G be a finite multiplicative group of order v , and D a k -subset of G . Then D is called a (v, k, λ) -difference set if any nonidentity element of G can be written in exactly λ ways as xy^{-1} where x and y are in D . We say that D is cyclic or abelian if G is.

Sometimes we want to be more specific and say that a difference set as above is a right difference set. If any nonidentity element of G can be written in exactly λ ways as $y^{-1}x$, then D is a left (v, k, λ) -difference set.

Definition 5.2. Let X be a subset of a finite group G . For any $g \in G$, define $Xg = \{xg : x \in X\}$. We call any set Xg a *right translate* of X .

We summarize some well-known results for difference sets.

Remark 5.3 (cf. Moore and Pollatsek [10]). Let D be a (v, k, λ) -difference set in a group G , then

- (1) the complement $\bar{D} = G \setminus D$ is a $(v, v - k, v - 2k + \lambda)$ -difference set,
- (2) for $g \in G$, both gD and Dg are (v, k, λ) -difference sets,
- (3) D is a left difference set if and only if it is a right difference set,
- (4) $\lambda(v - 1) = k(k - 1)$.

Here we assume that $v > 2k$. The case $v = 2k$ cannot occur because of the identity $\lambda(v - 1) = k(k - 1)$.

Construction 5.4 (Nilson and Cameron [11]). Let D be a nonempty proper subset of a finite group G . Construct an array $A(G, D)$ with rows indexed by D and columns by $G \setminus D$, with the (x, y) entry equal to $x^{-1}y$. Note that the entries of the array are nonidentity elements of G , and that no element of G is repeated in a row or column.

Definition 5.5. Let X be a subset of a group G and t an integer, then we define $X^{(t)} = \{x^t : x \in X\}$.

Proposition 5.6 (Nilson and Cameron [11]). *Suppose that D is a left difference set in a group G of order v . Then $A(G, D)$ is a double array of size $k \times (v - k)$ on $v - 1$ symbols. It is a triple array if and only if the size of $x^{-1}D \cap D^{(-1)}y$ is constant for $x \in D, y \in G \setminus D$. \square*

Remark 5.7. For double and triple arrays constructed from difference sets using Construction 5.4 we sometimes use the short notation DADS and TADS respectively.

Our main interest is triple arrays. In [11] it is proven when Construction 5.4 gives such an array, and here is a brief description of what is required. Let D be a difference set in a group G . An automorphism ϕ of G is called a *multiplier* of D if $\phi(D) = aDb$ for some $a, b \in G$. If G is abelian and ϕ is on the form $\phi_t : x \mapsto x^t$, then ϕ is called a *numerical multiplier* of D , but it is common practice to abuse terminology and call t itself a multiplier of D .

Theorem 5.8 (Nilson and Cameron [11]). *Let D be a (v, k, λ) -difference set in an abelian group G that admits -1 as a multiplier. Then there is a $k \times (v - k)$ triple array.*

In [11] also the converse is proved, that if A is a TADS from an difference set D in an abelian group G , then D admits -1 as a multiplier.

Definition 5.9. A difference set D is called *reversible* if $D = D^{(-1)}$.

If D is a difference set in an abelian group G , then -1 is a multiplier of D if and only if D has a reversible translate, as pointed out in [11].

In an nonabelian group G , -1 cannot be an automorphism, but -1 is called a *weak multiplier* of D if $D^{(-1)}$ is a translate Dg of D for some $g \in G$. However, it is an open problem if the construction can give a triple array from a nonabelian group, no such examples are known [11].

For convenience, we summarize and provide a translation table for group and array parameters as both come into play when working with DADS and TADS.

Remark 5.10. If A is a $TA(\omega, \kappa, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ constructed from a (v, k, λ) -difference set D by Construction 5.4, then $\omega = r + c - 1$ and

- (1) $\omega = v - 1,$
- (2) $\kappa = k - \lambda,$
- (3) $\lambda_{rr} = v - 2k + \lambda,$
- (4) $\lambda_{cc} = \lambda,$
- (5) $\lambda_{rc} = k - \lambda,$
- (6) $r = k,$
- (7) $c = v - k.$

The following observation will be used.

Lemma 5.11. *If A is an extremal $DA(\omega, \kappa, \lambda_{rr}, \lambda_{cc} : r \times c)$ then $\lambda_{cc}\lambda_{rr} = \kappa(\kappa - 1)$.*

Proof. That $\lambda_{cc} = r - \kappa$ and $\lambda_{rr} = c - \kappa$ is proven for extremal triple arrays in Lemma 2.1 in [7]. However, this also holds for extremal double arrays as pointed out here in Remark 2.8. We can calculate

$$\begin{aligned}\lambda_{cc}\lambda_{rr} &= (r - \kappa)(c - \kappa) = rc - r\kappa - c\kappa + \kappa^2 = \omega\kappa - r\kappa - c\kappa + \kappa^2 \\ &= \kappa(\omega - r - c + \kappa) = \kappa(r + c - 1 - r - c + \kappa) = \kappa(\kappa - 1).\end{aligned}\quad \square$$

5.2 | Results for double and triple arrays from difference sets

Lemma 5.12. *Let A be a double array constructed from a (v, k, λ) -difference set D in a group G by Construction 5.4. Then there is an intercalate, say in cells (x_1, y_1) , (x_2, y_2) , (x_2, y_1) and (x_1, y_2) if and only if there is an element $g \in G$ of order two such that*

$$g = x_1x_2^{-1} = x_2x_1^{-1} = y_1y_2^{-1} = y_2y_1^{-1}.$$

Proof. The cells (x_1, y_1) , (x_2, y_2) , (x_2, y_1) and (x_1, y_2) form an intercalate if and only if

$$\begin{cases} x_1^{-1}y_1 = x_2^{-1}y_2 \\ x_2^{-1}y_1 = x_1^{-1}y_2. \end{cases} \quad (1)$$

Eliminating y_2 from both equations in (1) gives

$$\begin{aligned}x_2x_1^{-1}y_1 &= x_1x_2^{-1}y_1 \\ x_2x_1^{-1} &= x_1x_2^{-1}.\end{aligned} \quad (2)$$

Also, eliminating x_1^{-1} from the first equation in (1) gives

$$\begin{aligned}y_1 &= x_1x_2^{-1}y_2 \\ y_1y_2^{-1} &= x_1x_2^{-1}\end{aligned}$$

and again from the first equation in (1) we have

$$\begin{aligned}x_1^{-1} &= x_2^{-1}y_2y_1^{-1} \\ x_2x_1^{-1} &= y_2y_1^{-1}.\end{aligned}$$

To see that g is of order two we use equation (2)

$$g^2 = (x_2x_1^{-1})^2 = x_2x_1^{-1}x_1x_2^{-1} = 1. \quad \square$$

Definition 5.13. In a group G we denote the subset of involutions, that is, the elements of order two by S . Note that the identity element of G is not in S .

Corollary 5.14. *Let A be a double array constructed from a (v, k, λ) -difference set D in a group G by Construction 5.4. Then A has intercalates if and only if $v = |G|$ is even.*

Proof. It is well known that if $|G|$ is even, then G has an odd number of involutions. It is also well known that the order of an element divides the order of the group, so if $|G|$ is odd then it has no elements of order two. □

Proposition 5.15. *Let A be a double array constructed from a (v, k, λ) -difference set D in a group G by Construction 5.4, then*

$$N(A) = \frac{\lambda(v - 2k + \lambda)|S|}{4}.$$

Proof. First, we note that the existence of elements of order two implies that v is even. This gives that both λ and $v - 2k + \lambda$ are even by the identity $\lambda(v - 1) = k(k - 1)$ for difference sets, so $N(A)$ is an integer. Let $s \in S$, then there are λ pairs $x_1, x_2 \in D$ with difference s . Since $x_1x_2^{-1} = x_2x_1^{-1}$, there are $\lambda/2$ pairs of rows with difference s in the array. For each of them there will be an intercalate when we have the same difference s between two columns in the array. Since $G \setminus D$ is a $(v, v - k, v - 2k + \lambda)$ -difference set, there will be $(v - 2k + \lambda)/2$ such pairs of columns y_1, y_2 . Hence, each $s \in S$ gives $\frac{\lambda}{2} \cdot \frac{v - 2k + \lambda}{2}$ intercalates and the result follows. □

Groups of order 16 are the smallest groups of even order having nontrivial difference sets. Kibler [4] gave a list of noncyclic (v, k, λ) -difference sets for $k < 20$. Here we find the smallest such group with $|S| = 1$, the generalized quaternion group of order 16, named (L) in [4]. It is nonabelian and has two nonisomorphic $(16, 6, 2)$ -difference sets, both giving proper $DA(15, 4, 6, 2 : 6 \times 10)$ with 3 intercalates. Cyclic groups of even order also have $|S| = 1$ and the smallest such group with a difference set is of order 40. Note that cyclic difference sets always give proper DADS, as pointed out in [11].

We have some special observations concerning abelian groups. Let G be an abelian group with $|S| > 1$, then $|G| \equiv 0 \pmod{4}$. This is because a group of even order has an odd number of elements of order 2. Further, if the group is abelian and has two elements of order 2, then it has a third element of order 2 such that these elements form together with the identity a subgroup of order 4, and the observation follows from Lagrange's Theorem.

Proposition 5.16. *Let A be a TADS constructed from an abelian difference set D , then A has intercalates.*

Proof. That A is a TADS means that D or a translate of D is reversible. In [3] it was proven that if D is a reversible (v, k, λ) -difference set in an abelian group, then v and λ are even, so $|S| > 0$ and the result follows. □

We summarize some well-known results for involutions in finite groups. Note that an abelian group in which every nontrivial element is of order p where p is a prime is called an elementary abelian p -group.

Remark 5.17 (cf. Miller [9], Wall [15]). Let G be a finite group, then

- (1) if G is an elementary abelian 2-group, then $|S| + 1 = |G|$.
- (2) if G is abelian but not an elementary abelian 2-group, then $|S| + 1 \leq \frac{|G|}{2}$.
- (3) if G is a nonabelian group, then $|S| + 1 \leq \frac{3|G|}{4}$.

In Figure 1 we saw a triple array A in which every two occurrences of an entry lie in an intercalate, that is, $\rho(A) = 1$. We now prove the existence of an infinite family of such triple arrays.

Theorem 5.18. *Let $m \geq 2$ be an integer, then there is a $TA(2^{2m} - 1, 2^{2m-2}, 2^{2m-2} + 2^m - 2^{m-1}, 2^{2m-2} - 2^{m-1}, 2^{2m-2} : (2^{2m-1} - 2^{m-1}) \times (2^{2m} - 2^{2m-1} + 2^{m-1}))$ in which every two occurrences of an entry lie in an intercalate. Furthermore, every TADS A with $\rho(A) = 1$ belongs to this family.*

Proof. Menon [8] proved that there for every integer $m > 1$ is a (v, k, λ) -difference set in elementary abelian 2-groups with $v = 2^{2m}, k = 2^{2m-1} - 2^{m-1}, \lambda = 2^{2m-2} - 2^{m-1}$. Since a difference set in an elementary abelian 2-group is reversible, Construction 5.4 gives a triple array [11]. As $|S| = |G| - 1 = \omega$ we have $\rho(A) = \frac{\kappa(\kappa-1)\omega}{4} \cdot \frac{4}{\kappa(\kappa-1)\omega} = 1$. Furthermore, Mann [5] proved that if there exists a nontrivial $2^m, k, \lambda$ configuration (difference set) then $k - \lambda = 2^{2s}, v = 2^{2s+2}$ from which we know that there are no other difference sets in elementary abelian 2-groups. The parameters are calculated using the identities in Remark 5.10. □

We look at some examples of interest and then calculate bounds for the number of intercalates in arrays from other abelian groups and from nonabelian groups.

Example 5.19. A triple array constructed by the $(16, 6, 2)$ -difference set 7 in the abelian group (C) in Kibler's list [4]. The group (C) is isomorphic to the direct product $\mathbb{Z}_4 \times V_4$ and has seven involutions (Figure 5).

- (C) Abelian $\langle a, b, c : a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle$
- 7. $\{1, a, a^2, ab, ac, a^3bc\}$

To make it easier to compare the number of intercalates, we use identities from Remark 5.10 and Lemma 5.11 to express $N(A)$ partly in array parameters.

abc	a^2b	a^2c	a^2bc	a^3	a^3b	a^3c	b	bc	c
bc	ab	ac	abc	a^2	a^2b	a^2c	a^3b	a^3bc	a^3c
a^3bc	b	c	bc	a	ab	ac	a^2b	a^2bc	a^2c
c	a	abc	ac	a^2b	a^2	a^2bc	a^3	a^3c	a^3bc
b	abc	a	ab	a^2c	a^2bc	a^2	a^3bc	a^3b	a^3
a^2	a^3c	a^3b	a^3	bc	c	b	ac	a	ab

FIGURE 5 A $TA(15, 4, 6, 2, 4 : 6 \times 10)$ with 21 intercalates

Remark 5.20. Let A be a $DADS(\omega, \kappa, \lambda_{rr}, \lambda_{cc} : r \times c)$ constructed from a (v, k, λ) -difference set in a group G , then

$$N(A) = \frac{\lambda_{cc}\lambda_{rr}|S|}{4} = \frac{\kappa(\kappa - 1)|S|}{4}.$$

Proposition 5.21. *Let A be a $DADS(\omega, \kappa, \lambda_{rr}, \lambda_{cc} : r \times c)$ constructed from an abelian group G which is not an elementary abelian 2-group, then*

$$N(A) \leq \frac{\kappa(\kappa - 1)(\omega - 1)}{8},$$

and the bound is sharp.

Proof. From Remark 5.17 we have $|S| \leq \frac{|G|-2}{2} = \frac{\omega-1}{2}$, which inserted in Proposition 5.15 gives $N(A) \leq \frac{\kappa(\kappa-1)(\omega-1)}{8}$. By the triple array in Figure 5 we know that the bound is sharp. □

Example 5.22. A proper double array constructed by difference set 9 in the nonabelian group (E) in Kibler's list [4]. The group (E) is isomorphic to the direct product $D_8 \times \mathbb{Z}_2$ and has 11 involutions (Figure 6).

(E) Nonabelian $\langle a, b, c : a^4 = b^2 = c^2 = 1, bab = a^3, ac = ca, bc = cb \rangle$
 9. $\{1, a, a^2, b, ac, a^2bc\}$

Proposition 5.23. *Let A be a $DADS(\omega, \kappa, \lambda_{rr}, \lambda_{cc} : r \times c)$ constructed from a nonabelian group G , then*

$$N(A) \leq \frac{\kappa(\kappa - 1)(3\omega - 1)}{16}.$$

and the bound is sharp.

Proof. From Remark 5.17 we have $|S| \leq \frac{3|G|-4}{4} = \frac{3(\omega+1)-4}{4} = \frac{3\omega-1}{4}$. which inserted in Proposition 5.15 gives $N(A) \leq \frac{\kappa(\kappa-1)(3\omega-1)}{16}$. By the proper double array in Figure 6 we know that the bound is sharp. □

ab	abc	a^2b	a^2c	a^3	a^3b	a^3c	a^3bc	bc	c
b	bc	ab	ac	a^2	a^2b	a^2c	a^2bc	a^3bc	a^3c
a^3b	a^3bc	b	c	a	ab	ac	abc	a^2bc	a^2c
a^3	a^3c	a^2	a^2bc	ab	a	abc	ac	c	bc
bc	b	abc	a	a^2c	a^2bc	a^2	a^2b	a^3b	a^3
ac	a	c	b	a^3bc	a^3c	a^3b	a^3	a^2	a^2b

FIGURE 6 A proper $DA(15, 4, 6, 2 : 6 \times 10)$ with 33 intercalates

The next result simplifies the calculations when determining the density of intercalates in DADS. It follows directly from the definition and Remark 5.20.

Corollary 5.24. *Let A be a DADS on ω symbols constructed from a group with $|S|$ involutions, then $\rho(A) = \frac{|S|}{\omega}$.*

Now we determine the maximal density for arrays coming from groups other than the elementary abelian 2-groups.

Corollary 5.25. *Let A be a DADS $(\omega, \kappa, \lambda_{rr}, \lambda_{cc} : r \times c)$ constructed from a group G , then*

- (1) *if G is abelian, but not an elementary abelian 2-group, then $\rho(A) \leq \frac{\omega-1}{2\omega}$,*
- (2) *if G is nonabelian, then $\rho(A) \leq \frac{3\omega-1}{4\omega}$.*

Proof. We take $|S|$ from Propositions 5.21 and 5.23, respectively, and use Lemma 5.24. If G is abelian we get

$$\rho(A) \leq \frac{\omega-1}{2} \cdot \frac{1}{\omega} = \frac{\omega-1}{2\omega}$$

and if G is nonabelian we get

$$\rho(A) \leq \frac{3\omega-1}{4} \cdot \frac{1}{\omega} = \frac{3\omega-1}{4\omega}.$$

□

We note that a DADS A from an abelian group which is not an elementary 2-group has a maximal value $\rho(A) = \frac{7}{15}$ as we have in Example 5.19. If A is constructed from a nonabelian group then it has a maximal value $\rho(A) = \frac{11}{15}$ as in Example 5.22.

6 | CONCLUDING REMARKS

In conclusion, we pose the following problems.

Problem 6.1. For most types of arrays here, the maximal density decreases with the size of the array. The exceptions are TADS with full density and PTAs where $q = r \equiv 3 \pmod{4}$, which could potentially have high density. Are there large examples of PTAs with almost full density of intercalates?

Problem 6.2. To further the investigation of properties of double and triple arrays, check for the presence of generalized intercalates, that is, embedded $m \times n$ Latin rectangles.

Problem 6.3. For DADS and TADS there are different bounds for the maximal density depending on if the underlying group is abelian or not. The higher bound is for nonabelian groups. Is it possible to construct TADS from nonabelian groups?

Problem 6.4. Prove or disprove Agrawal's Conjecture.

ORCID

Tomas Nilson  <http://orcid.org/0000-0003-0930-7116>

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