Further Investigations of Convergence Results for Homogenization Problems with Various Combinations of Scales

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I would like to dedicate this thesis to my two miracles:
Michail, Alexei...
and to my husband Mats-Erik

I'm fixing a hole where the rain gets in
And stops my mind from wandering
Where it will go…

- John Lennon and Paul McCartney
´´Fixing a Hole´´, The Beatles.

List of appended papers

This thesis is based primarily on the following six papers, herein referred to by their Roman numerals:

Paper I.
Homogenization of linear parabolic equations with a certain resonant matching between rapid spatial and temporal oscillations in periodically perforated domains.
T. Lobkova.
Acta Math Sin Engl Ser. 35 (2) (2019), 340–358.

Paper II.
Homogenization of a linear parabolic problem with a certain type of matching between the microscopic scales.
P. Johnsen, T. Lobkova.
Appl. Math. 63 (2018), no. 5, 503–521.

Paper III.
Homogenization of the heat equation with a vanishing volumetric heat capacity. (2019)
T. **Danielsson**[1], P. Johnsen.
In: Faragó I., Izsák F., Simon P. (eds) Progress in Industrial Mathematics at ECMI 2018. Mathematics in Industry, vol 30. Springer, Cham.

[1]*Last name changed to Danielsson from Lobkova in May 2019.*

Paper IV.
Homogenization of linear parabolic equations with three spatial and three temporal microscopic scales for certain matching between the microscopic scales.
T. Danielsson, P. Johnsen.
Submitted for publication in *Mathematica Bohemica*.
The arXiv e-print version, 2019, arXiv: 1908.05892 [math.AP].

Paper V.
Homogenization of monotone parabolic problems with an arbitrary number of spatial and temporal scales.
T. Danielsson, L. Flodén, P. Johnsen, M. Olsson Lindberg.
Submitted for publication in *Pure and Applied Mathematics Quarterly*.
The arXiv e-print version, 2020, arXiv: 1704.01375v3 [math.AP].

Paper VI.
On some concepts of convergence and their connections.
T. Danielsson, L. Flodén, A. Holmbom, P. Johnsen, M. Olsson Lindberg.
Manuscript.

The published papers were included with permissions from the publishers.

Author contributions

My contribution to the papers is listed below:

Paper I.

The author of this paper is the sole author. The author carried out a literature review and conducted the research.

Paper II.

This is a collaborative work. The author took part in the development of some of the main results. The most important contributions by the author are found in Section 2. She also took part in the development of the homogenization procedure in Section 3.

Paper III.

This is a collaborative work. The author took part in the theoretical framework and worked on the referee's suggestions.

Paper IV.

This is a collaborative work. The author contributed with preparatory work and theoretical development of separate sections with feedback from the co-author.

Paper V.

This is a collaborative work. The author played a prominent role in the development of the main result of the paper, found in Section 3, and in writing the paper. The author also took part in forming the example in Section 4.

Paper IV.

The article is a collaborative work. The author took part in the development of both Section 5 and subsection 6.1. Additionally, the author was responsible for working out the proof in Section 6.1.

Further Investigations of Convergence Results for Homogenization Problems with Various Combinations of Scales

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Abstract

This thesis is based on six papers. We study the homogenization of selected parabolic problems with one or more microscopic scales in space and time, respectively. The approaches are prepared by means of certain methods, like two-scale convergence, multiscale convergence and also the evolution setting of multiscale convergence and very weak multiscale convergence. Paper I treats a linear parabolic homogenization problem with rapid spatial and temporal oscillations in perforated domains. Suitable results of two-scale convergence type are established. Paper II deals with further development of compactness results which can be used in the homogenization procedure engaging a certain limit condition. The homogenization procedure deals with a parabolic problem with a certain matching between a fast spatial and a fast temporal scale and a coefficient passing to zero that the time derivative is multiplied with. Papers III and IV are further generalizations of Paper II and investigate homogenization problems with different types of matching between the microscopic scales. Papers III and IV deal with one and two rapid scales in both space and time respectively. Paper V treats the nonlinearity of monotone parabolic problems with an arbitrary number of spatial and temporal scales by applying the perturbed test functions method together with multiscale convergence and very weak multiscale convergence adapted to the evolution setting. In Paper VI we discuss the relation between two-scale convergence and the unfolding method and potential extensions of existing results. The papers above are summarized in Chapter 4. Chapter 1 gives a brief introduction to the topic and Chapters 2 and 3 are surveys over some important previous results

Acknowledgements

The work presented in this thesis has been completed at the Department of Mathematics and Science Education at the Mid Sweden University. I gratefully acknowledge the Mid Sweden University for the financial support I received throughout these years. There are many people who in one way or another have contributed to the fact that I have been able to write this thesis.

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Thank you, mom and dad, for your encouragement over the years. And my in-laws, Eva and Kjell, for helping in whatever way you could during this most demanding period. A special thanks goes to Anna Holmstedt for her friendship since that fateful spring of 2013.

Last but not least, I am grateful to my husband Mats-Erik for your never-ending source of strength and inspiration through all these years. To my sons, Michail and Alexei, who make me remember the essentials of life.

Tatiana Danielsson.

Östersund, Mars 2020.

Notation

The following symbols and sets are used in the thesis.

X : Any linear space.
X' : The dual space of X.
$\|u\|_X$: The norm of $u \in X$, when X is a normed space.
$\{\varepsilon\}$: A sequence $\{\varepsilon(h)\}$ such that $\varepsilon = \varepsilon(h) \to 0$ as $h \to \infty$.
$\{u_h\}$: A sequence of functions u_h.
$u_h \to u$: $\{u_h\}$ converges strongly to u.
$u_h \rightharpoonup u$: $\{u_h\}$ converges weakly to u.
Ω : Open bounded subset of $\mathbb{R}^N$ with a smooth boundary.
$\partial\Omega$: The boundary of Ω.
Ω_T : The set $\Omega \times (0,T)$.
Ω_ε : A domain with small identical holes situated periodically with period ε in Ω.
Y : The unit cube $(0,1)^N$.
$Y^n = Y_1 \times \ldots \times Y_n$, where $Y_j = Y$, $j = 1, \ldots, n$.
Y^* : An open subset of Y such that $E_\sharp(Y^*)$ is smooth and connected.
$E_\sharp(Y^*)$: The Y^*-periodic extension of Y^* infinitely along all principal directions of $\mathbb{R}^N$.
S : The interval $(0,1)$.
$S^m = S_1 \times \ldots \times S_m$, where $S_j = S$, $j = 1, \ldots, m$.
$\mathcal{Y}_{n,m}$: The set $Y^n \times S^m$.

Below is a list of function spaces.

$G(A)$: A space of real valued functions defined on A.
$G(A)/\mathbb{R}$: The space $\left\{u \in G(A) \,\middle|\, \int_A u(y)dy = 0\right\}$.
$D(\Omega)$: The space of $C^\infty(\Omega)$-functions with compact support in Ω.
$C^\infty_\sharp(Y)$: The space of Y-periodic functions in $C^\infty(\mathbb{R}^N)$.
$H^1_\sharp(Y)$: The closure of $C^\infty_\sharp(Y)$ with respect to the $H^1(Y)$-norm.
$C^\infty_\sharp(Y^*)$: $C^\infty(E_\sharp(Y^*))$-functions that are periodic with respect to Y^*.
$D_\sharp(Y^*)$: The functions in $C^\infty_\sharp(Y^*)$ with support contained in $E_\sharp(Y^*)$.
$H^1_\sharp(Y^*)$: $H^1_{loc}(E_\sharp(Y^*))$-functions that are periodic with respect to Y^*.
$L^p(a,b;G(A))$: The space $\left\{u : (a,b) \to G(A) \mid \int_a^b \|u\|^p_{G(A)}\, dt < \infty\right\}$.
$L^\infty(a,b;G(A))$: The space $\left\{u : (a,b) \to G(A) \mid \operatorname*{ess\,sup}_{t\in(a,b)} \|u\|_{G(A)} < \infty\right\}$.
$D(B;G(A))$: The space of infinitely differentiable functions $\{u \mid u : B \to G(A)\}$ with compact support in B.
$W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$: The space $\left\{u \,|\, u \in L^2(0,T;H_0^1(\Omega))\right.$ and $\left.\partial_t u \in L^2(0,T;H^{-1}(\Omega))\right\}$ with the norm
$\|u\|_{W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))} = \|u\|_{L^2(0,T;H_0^1(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^{-1}(\Omega))}$
$\mathcal{W} = \left\{z \in L^2_\sharp(S;H^1_\sharp(Y)/\mathbb{R}) : \partial_s z \in L^2_\sharp(S;\left(H^1_\sharp(Y)/\mathbb{R}\right)')\right\}$
$\mathcal{W}_{i,j} = \left\{z \in L^2_\sharp(S_j;H^1_\sharp(Y_i)/\mathbb{R}) : \partial_{s_j} z \in L^2_\sharp(S_j;\left(H^1_\sharp(Y_i)/\mathbb{R}\right)')\right\}$

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1 Introduction

Engineering materials, to some degree, can be sorted into four essential groups: metals, ceramics, polymers and composites. Composite materials are characterized by combining two or more materials that have substantially different types of properties in order to achieve a new set of properties. Some common composites are reinforced concrete, fibre-reinforced polymers, carbon fibre and engineered wood. Some composites can be described by two or more scales, the microscopic length scales which are associated with the heterogeneities and the macroscopic scale at which we perceive a sample as homogeneous. From the mathematical point of view, various physical mechanisms in composite materials can be described by partial differential equations with highly oscillating coefficients in one or several microscopic scales.

Other types of inhomogeneous materials are those which contain some sort of voids. Some well-known examples of perforated media are plastic films, punched plate, skin and bones. Working with perforated materials, physical phenomena can be described by boundary value problems defined on domains with holes.

Handling the challenges which occur while trying to treat the mathematical models mentioned above numerically is often difficult or even impossible. An alternative way is to use homogenization. The aim of homogenization is to find the effective characteristics and construct a homogenized problem. To be able to find these effective properties, one has to take into account the heterogenous structure. The idea is to imagine that the microstructure becomes finer and finer in order to in the limit case find the property of a corresponding homogeneous material. By using a certain type of microscopic description as a starting point, we go on and seek a macroscopic model of the problem.

Nowadays, there are many engineers and material scientists who are engaged in constructing composite materials with certain desired properties. For example, getting different mixtures of stiffness and softness or high stiffness and low weight composites can be of high interest. A natural question arises: how should we mix the materials to get the desired properties? Homogenization can be useful to predict these properties.

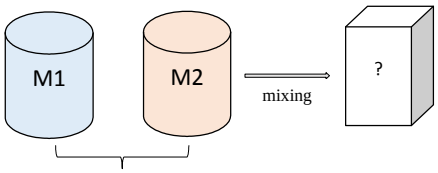


Figure1. *Will the mixed material have the desired properties?*

It turns out that the effective property is not straightforward to determine but can be found as a particular sort of averaging of the properties of the constituents.

1.1 Homogenization background

To illustrate the ideas of homogenization theory we shall begin by discussing the following model problem of conductivity. Our equation reads

$$\begin{cases} -\nabla \cdot \left(A\left(\frac{x}{\varepsilon}\right) \nabla u_\varepsilon(x) \right) &= f(x) & \text{in } \Omega, \\ u_\varepsilon(x) &= 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^N$ is an open bounded domain with smooth boundary, which represents a material with periodic heterogeneities of length-scale ε described by a periodic heat conductivity matrix $A(y)$ with period $Y = (0,1)^N$. This means that $A\left(\frac{x}{\varepsilon}\right)$, $x \in \Omega$, is periodic with period $(0,\varepsilon)^N$. By f we denote the source of heat inside the material and u_ε represents the temperature distribution in Ω and equals 0 on the surface $\partial\Omega$ of the body. Equation (1) represents a classical elliptic boundary value problem, which allows a unique solution u_ε, under certain requirements on A, f and $\partial\Omega$.

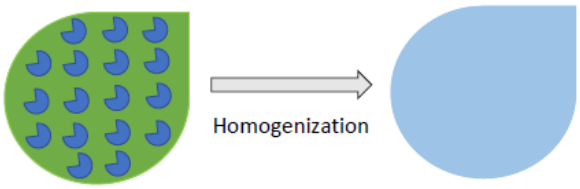


Figure 2. *Homogenizing the mixture.*

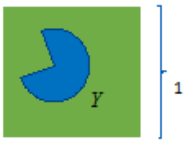


Figure 3. *The representative cell upscaled to a unit cell.*

Usually, considerable effort is required to solve the problem numerically. This is because ε, the scale of the microstructure, is small and the number of degrees of freedom grows rapidly as ε tends to zero. Observing that even though the material is heterogeneous, it will behave as a homogeneous one on a macroscopical level. This means that an alternative is to try to find the macroscopic or homogenized behaviour of the material contained in Ω and find an approximation of u_ε.

The main feature of the homogenization process consists of studying a sequence of problems using a small parameter ε ($\varepsilon > 0$) that goes to zero. The key is to find a limit u to the sequence $\{u_\varepsilon\}$ and determining the problem, which u solves. A difficulty is to decide what type of convergence to use as ε goes to zero. Here $\{u_\varepsilon\}$ goes weakly in $H_0^1(\Omega)$ to u, which is the solution to the limit equation

$$\begin{cases} -\nabla \cdot (B \nabla u(x)) &= f(x) & \text{in } \Omega, \\ u(x) &= 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where B is calculated by a so-called local problem defined on a representative unit upscaled to a unit cell; see Figures 2 and 3. Here, the homogenized matrix B defines the effective heat conductivity.

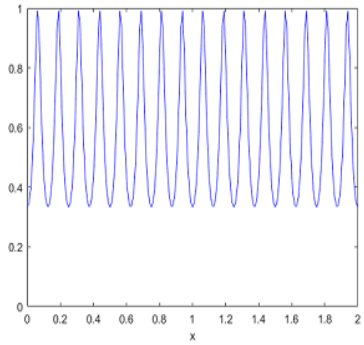
We shall end this section by demonstrating a particular one-dimensional example of (1). More precisely, we let $\Omega = (0, 2)$ and study

$$\begin{cases} -\frac{d}{dx}\left(A\left(\frac{x}{\varepsilon}\right)\frac{d}{dx}u_\varepsilon(x)\right) &= f(x) \quad \text{in } \Omega, \\ u_\varepsilon(0) = u_\varepsilon(2) &= 0, \end{cases} \tag{3}$$

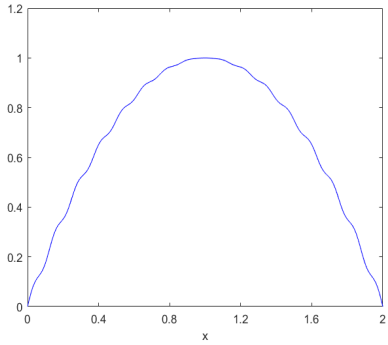
where we choose the function $A(y)$ and the right-hand side function $f(x)$ as

$$A(y) = (2 + \cos(2\pi y))^{-1}, \quad f(x) = 1.$$

As $\varepsilon \to 0$, our function $A\left(\frac{x}{\varepsilon}\right)$ is oscillating more and more. We plot the coefficient $A\left(\frac{x}{\varepsilon}\right)$ and the corresponding solution $u_\varepsilon(x)$ of (3) in Figure 4.



a) The coefficient $A\left(\frac{x}{\varepsilon}\right)$, for $\varepsilon = 2^{-3}$.



b) The corresponding solution $u_\varepsilon(x)$ of (1).

Figure 4. *Illustration of equation (3).*

A natural next question is to find ways to compute B. Here the periodic character of $A\left(\frac{x}{\varepsilon}\right)$ is helpful.

If $\varepsilon \to 0$, by periodicity, we have

$$A\left(\frac{x}{\varepsilon}\right) \rightharpoonup \widetilde{A} = \int_Y A(y)dy \text{ in } L^2(\Omega)^{N\times N},$$

see Theorem 2.8 in [11]. Here, $\widetilde{A}$ is a reasonable guess for the matrix in (2) as it is both the weak limit of $A\left(\frac{x}{\varepsilon}\right)$ and the arithmetic mean of A.

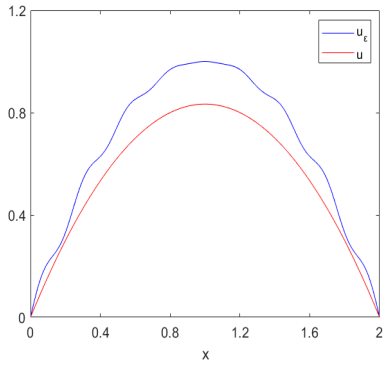


Figure 5. u_ε *and* u, *for* $\varepsilon = 2^{-2}$.

From Figure 5 we can see that our solution u does not approximate the exact solution u_ε very well and, thus, does not give the correct homogenized solution. Obviously, $\widetilde{A}$ is not the appropriate choice of B and our guess is wrong. Below, we show Figures 6 and 7, where thin layers with strong heat-insulating material are inserted.

Figure 6. $N = 1$.

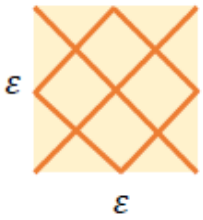


Figure 7. $N = 2$.

These types of layers affect the heat conduction greatly, but very little the arithmetic mean $\widetilde{A}$, which explains why the guess failed.

The coefficient B that governs our limit equation must be determined differently. In our case,

$$u_\varepsilon \rightharpoonup u \text{ in } H_0^1(\Omega),$$

where u is the solution to the following limit problem

$$\begin{cases} -\frac{d}{dx}\left(B\frac{d}{dx}u\left(x\right)\right) &= f(x) \quad \text{in } \Omega, \\ u(0) = u(2) &= 0, \end{cases}$$

when B is given by

$$B = \left(\int_Y A^{-1}(y)dy\right)^{-1}, \tag{4}$$

i.e. the harmonic mean of A over Y. See Theorem 5.5 in [11]. See also [22].

Remark 1 *Note, that (4) only applies to the one-dimensional case. There is no equivalent to the harmonic mean for problems with higher dimensions. However, some bounds on the homogenized coefficients can be obtained in more than one-dimensional cases. See Chapter 2 in [5], where particular calculation formulas are also available for special geometries.*

Based on our choice of A, we have from (4) the homogenized coefficient

$$B = \left(\int_0^1 \left((2 + \cos(2\pi y))^{-1} \right)^{-1} dy \right)^{-1} = \left(\int_0^1 (2 + \cos(2\pi y))\, dy \right)^{-1} = \frac{1}{2}.$$

In figures $8-10$, we show the approximate solution u together with the exact solution u_ε. We get the following outcome by choosing $\varepsilon = 2^{-1}$.

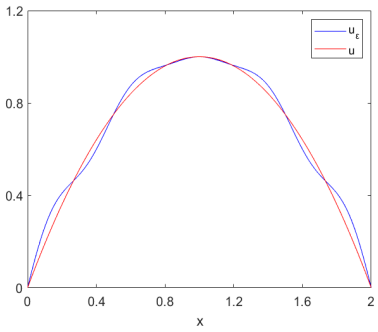


Figure 8. u_ε *and* u, *for* $\varepsilon = 2^{-1}$.

The approximation progresses as we let $\varepsilon = 2^{-2}$.

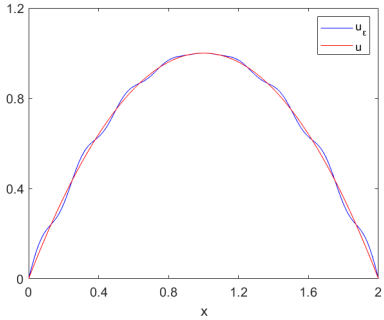


Figure 9. u_ε *and* u, *for* $\varepsilon = 2^{-2}$.

For $\varepsilon = 2^{-3}$, the approximation gets even better.

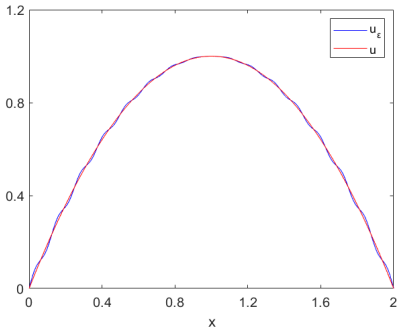


Figure 10. u_ε *and* u, *for* $\varepsilon = 2^{-3}$.

In higher dimensions the problem of finding the homogenized coefficient B is treated by solving certain differential equations, so-called local problems or cell problems, defined on the representative unit Y.

1.2 Aim and research questions

The overall purpose of this thesis is to further develop techniques that are applicable in homogenization theory and apply them to some homogenization problems. It addresses the study of two-scale convergence and its generalization adapted to evolution problems with microoscillations in various combinations of scales. We also study homogenization problems defined on periodically perforated domains and develop techniques suitable for such problems. To accomplish the stated aim, we identified the following sub-aims:

1. Develop techniques without nontrivial extensions suitable for certain evolution problems in perforated domains;

2. Prove compactness results for the evolution setting of multiscale and very weak multiscale convergence with another assumption than the traditional one and investigate new applications to homogenization problems;

3. Examine possible extensions of existing results ([28], [30]) for certain parabolic problems with multiple spatial and temporal scales beyond the linear setting;

4. Discuss some essential types of convergence such as two-scale convergence and unfolding and their relationship to each other.

The respective papers address the research questions given below.

Paper I. How should the techniques developed for homogenization in periodically perforated domains in [2] be generalized to be applicable to certain evolution problems with rapid scales of oscillations in both space and time?

Paper II. Does the appearance of the coefficient ε in front of the time derivative give rise to some new unexpected results for a linear parabolic problem with

a certain matching between the frequencies of oscillation in the rapid spatial and temporal scales?

Paper III. Does the relationship between the coefficient ε^p in front of the time derivative and the frequencies of the spatial and temporal microoscillations in the parabolic equation affect the homogenized problem and its associated local problem?

Paper IV. Which outcome would one get after the generalization of the results presented in Paper III to several spatial and temporal scales?

Paper V. How do we design a homogenization procedure for parabolic problems exhibiting arbitrary numbers of scales in time and space if the scale functions are not necessarily powers of ε and the oscillating operator is monotone but not necessarily linear?

Paper VI. Will the comparison between the essential properties of two-scale convergence and unfolding lead us to new conclusions?

Taking into account all sub-aims we can sum up that Paper I addresses 1, Papers II, III and IV are responsible for completing aim 2 while Papers V and VI address 3 and 4, respectively.

1.3 Outline of the thesis

The thesis is organized as follows: Chapters 2 and 3 are surveys over important existing results, while Chapter 4 summarizes the new contributions of the thesis. Chapter 2 is devoted to convergence modes for functions and operators. In Section 2.1 we formulate some classical results concerning G-convergence for both elliptic and parabolic operators. We also include G-convergence for monotone operators. In Section 2.2, two-scale convergence and its generalizations are described by given definitions, theorems and illustrated examples. Section 2.3 concerns a basic introduction of the periodic unfolding method.

In Chapter 3 we consider homogenization problems in perforated domains by presenting the main issues of such problems and the ways of finding the homogenized problem. In Section 3.1 we briefly describe extension techniques suitable for sequences of solutions to linear elliptic problems with mixed boundary conditions in perforated domains. Section 3.2 deals with homogenization in perforated domains with homogenous Dirichlet data on the boundary of the holes and the connection to porous media. In subsection 3.2.1 we outline different limit problems of the Poisson equation in a periodically perforated domain with homogenous Dirichlet data on the boundary of the pores. The limit cases look different depending on how the perforation's proportion of the repetitive unit develops during the limit process. In subsection 3.2.2 we study the homogenization of the Stokes equation in porous media. Here, the perforation's proportion of the repetitive unit is constant during the limit process. The convergence of the homogenization process is presented by Theorem 46, providing unique solutions for the limits of $\{u_\varepsilon\}$ and $\{p_\varepsilon\}$.

In Chapter 4 the results from the papers are presented as a summary of each of them.

In Chapter 5 we discuss the thesis outcome based on the findings of our research.

Finally, we attached the appended papers or links to the papers.

2 Convergence modes related to homogenization

We shall start this chapter by recalling some main results about G-convergence for linear elliptic and parabolic operators as well as giving an overview of the monotone case. Then we will move on to describe two-scale convergence, which will be followed by a presentation of multiscale and evolution multiscale convergence, respectively. The chapter ends with an introduction of the periodic unfolding method and its connection to two-scale convergence.

2.1 G-convergence

G-convergence was first developed in the late sixties by Spagnolo in his paper [53], to express the convergence of sequences of linear elliptic and parabolic partial differential operators. This concept of convergence was further developed by Murat and Tartar in their papers [40] and [41]. Some important results on G-convergence are also proven in [56], [17] and [46]. To make it more informative let us take a closer look at the operator

$$\mathcal{A}\colon X \to X',$$

where X' is the dual space of X. If $\mathcal{A}$ has appropriate properties then there is a unique solution $u \in X$ to

$$\mathcal{A}u = f,$$

where $f \in X'$.

Let us now consider a sequence of equations

$$\mathcal{A}_h u_h = f,$$

where $\{\mathcal{A}_h\}$ is a sequence of operators and $\{u_h\}$ the corresponding sequence of unique solutions. If u_h approaches $u \in X$, the unique solution to

$$\mathcal{B}u = f,$$

when $h \to \infty$ and $\mathcal{B}$ is the same for any $f \in X'$, we can in some sense say that $\{\mathcal{A}_h\}$ converges to $\mathcal{B}$. Compared to periodic homogenization, G-convergence is a more general concept and does not include any technique for calculating the coefficient in the limit operator.

In the following two subsections, we shall show four well-known types of G-convergence: linear elliptic and parabolic, and monotone elliptic and parabolic.

2.1.1 Elliptic and parabolic G-convergence for linear operators

We shall initiate this subsection by defining the notion of G-convergence for linear elliptic and parabolic operators as well as introducing some important properties of the G-limit. We shall also give compactness results for both cases.

We consider a sequence $\{A_h\}$ and we let $f \in H^{-1}(\Omega)$, where Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary. For every fixed h there exists a unique solution to the Dirichlet boundary value problem

$$\begin{array}{rcll} -\nabla \cdot (A_h(x)\nabla u_h(x)) & = & f(x) & \text{in } \Omega, \\ u_h(x) & = & 0 & \text{on } \partial\Omega, \end{array} \tag{5}$$

if A_h obeys certain conditions.
Here A_h belongs to $N(\lambda, \mu, \Omega)$, which denotes the set of all functions

$$A : \Omega \to \mathbb{R}^{N^2},$$

satisfying the following properties:
$(e1)$ $A \in L^\infty(\Omega)^{N\times N}$;
$(e2)$ $A(x)\xi \cdot \xi \geqslant \lambda |\xi|^2$ a.e. in Ω and for all $\xi \in R^N$;
$(e3)$ $|A(x)\xi| \leqslant \mu |\xi|$ a.e. in Ω and for all $\xi \in R^N$,
where λ and μ are two positive constants such that $0 < \lambda \leq \mu < +\infty$.
We introduce the following notion.

Definition 2 *The sequence $\{A_h\}$ of symmetric matrices in $N(\lambda, \mu, \Omega)$ is said to G-converge to $B \in N(\lambda', \mu', \Omega)$ if, for every $f \in H^{-1}(\Omega)$, the sequence of solutions in (5) satisfies*

$$u_h(x) \rightharpoonup u(x) \text{ in } H_0^1(\Omega)$$

and

$$A_h(x)\nabla u_h(x) \rightharpoonup B(x)\nabla u(x) \text{ in } L^2(\Omega)^N, \tag{6}$$

where u is the unique solution to

$$\left\{ \begin{array}{rcll} -\nabla \cdot (B(x)\nabla u(x)) & = & f(x) & \text{in } \Omega, \\ u(x) & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

The following compactness result holds true:

Theorem 3 *Suppose that the sequence $\{A_h\}$ of the symmetric matrices belongs to $N(\lambda, \mu, \Omega)$. Then there is a subsequence that G-converges to a limit B in the same set.*

Proof. See Proposition 3 in [53]. ■

Remark 4 *The condition (6) is not needed for symmetric matrices A_h, but necessary for the non-symmetric case. See Remark 7.2 in [22].*

The following holds without assuming symmetry.

Theorem 5 *Let $\{A_h\}$ be a sequence such that $A_h \in N(\lambda, \mu, \Omega)$. Then there exists a subsequence that G-converges to some $B \in N\left(\lambda, \frac{\mu^2}{\lambda}, \Omega\right)$.*

Proof. See Theorem 2 in [41]. ■

We give some theorems concerning important properties of the G-limit.

Theorem 6 *The G-limit does not depend on the boundary conditions.*

Proof. See Proposition 1.2.19 in [5]. ■

Theorem 7 *The G-limit is unique.*

Proof. See Proposition 1.2.18 in [5] and Proposition 7.3 in [22]. ■

Our next concern is parabolic problems. Results on G-convergence for linear parabolic problems are given in [52], [53] and [54]. Moreover, some properties were studied by Colombini and Spagnolo in [20].

We study a parabolic problem of the type

$$\left\{\begin{array}{rcll} \partial_t u_h(x,t) - \nabla\cdot(A_h(x,t)\nabla u_h(x,t)) & = & f(x,t) & \text{in } \Omega_T, \\ u_h(x,0) & = & u^0(x) & \text{in } \Omega, \\ u_h(x,t) & = & 0 & \text{on } \partial\Omega\times(0,T), \end{array}\right. \tag{7}$$

where $f \in (0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$.

Let $S(\lambda,\mu,\Omega_T)$ denote the set of all functions

$$A : \Omega_T \to \mathbb{R}^{N^2},$$

which satisfy the following conditions:
$(p1)$ $A \in L^\infty(\Omega_T)^{N\times N}$;
$(p2)$ $A(x,t)\xi\cdot\xi \geqslant \lambda|\xi|^2$ a.e. in Ω_T and for all $\xi \in R^N$;
$(p3)$ $|A(x,t)\xi| \leqslant \mu|\xi|$ a.e. in Ω_T and for all $\xi \in R^N$,
where λ and μ are constants, such that $0 < \lambda \leqslant \mu < +\infty$.

We define G-convergence for parabolic operators in the following way.

Definition 8 *Let A_h belong to $S(\lambda,\mu,\Omega_T)$. A sequence $\{A_h\}$ is said to G-converge to $B \in S(\lambda',\mu',\Omega_T)$ if, for every $f \in L^2(0,T;H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, the sequence $\{u_h\}$ of solutions to the equations (7) satisfies*

$$\begin{array}{rcll} u_h(x,t) & \rightharpoonup & u(x,t) & \text{in } L^2(0,T;H_0^1(\Omega)), \\ A_h(x,t)\nabla u_h(x,t) & \rightharpoonup & B(x,t)\nabla u(x,t) & \text{in } L^2(\Omega_T)^N, \end{array}$$

where u is the unique solution to

$$\begin{array}{rcll} \partial_t u(x,t) - \nabla\cdot(B(x,t)\nabla u(x,t)) & = & f(x,t) & \text{in } \Omega_T, \\ u(x,0) & = & u^0(x) & \text{in } \Omega, \\ u(x,t) & = & 0 & \text{on } \partial\Omega\times(0,T). \end{array}$$

The next theorem presents a compactness result for the parabolic case.

Theorem 9 *Let $\{A_h\}$ be a sequence such that $A_h \in S(\lambda,\mu,\Omega_T)$. Then there exists a subsequence that G-converges to some $B \in S\left(\lambda,\frac{\mu^2}{\lambda},\Omega_T\right)$.*

Proof. See Chapter 3 in [54] and Theorem 3.1 in [55]. ■

2.1.2 Elliptic and parabolic G-convergence for monotone operators

In this subsection we shall define G-convergence for both monotone elliptic and monotone parabolic operators. G-convergence can be applied to nonlinear monotone elliptic equations under certain assumptions. To be able to keep the sequence of operators under control we have to impose suitable monotonicity and continuity conditions. Many have studied these types of problems. We give a few examples. Tartar was the first to investigate the nonlinear case, see [57] and [58]. G-convergence for sequences of maximal monotone operators was presented by Chiado' Piat, Dal Maso and Defranceschi in [17]. A study of some classes of strictly monotone operators is conducted by Raitum in [50].

Let us study

$$\begin{aligned} -\nabla \cdot A_h(x, \nabla u_h(x)) &= f(x) \quad \text{in } \Omega, \\ u_h(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{8}$$

where $f \in H^{-1}(\Omega)$ and $u_h \in H_0^1(\Omega)$.

We need some conditions on A_h. Let $M(\lambda, \mu, \beta, \Omega)$ be the set of all functions

$$A : \Omega \times R^N \to R^N,$$

satisfying the following:

$(M1)$ $A(\cdot, \xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^N$;
$(M2)$ $A(x, 0) = 0$ a.e. in Ω;
$(M3)$ $(A(x, \xi_1) - A(x, \xi_2)) \cdot (\xi_1 - \xi_2) \geq \lambda |\xi_1 - \xi_2|^2$ a.e. in Ω for all $\xi_1, \xi_2 \in \mathbb{R}^N$;
$(M4)$ $|A(x, \xi_1) - A(x, \xi_2)| \leq \mu(1 + |\xi_1| + |\xi_2|)^{1-\beta} |\xi_1 - \xi_2|^\beta$ a.e. in Ω for all $\xi_1, \xi_2 \in \mathbb{R}^N$,
where $\lambda, \mu > 0$ and $0 < \beta \leq 1$.

Next, we establish the following definition.

Definition 10 *The sequence $\{A_h\} \in M(\lambda, \mu, \beta, \Omega)$ is said to G-converge to $B \in M(\lambda', \mu', \beta', \Omega)$ if, for any $f \in H^{-1}(\Omega)$, the solutions u_h to the sequence of problems (8) satisfy*

$$\begin{aligned} u_h(x) &\rightharpoonup u(x) && \text{in } H_0^1(\Omega), \\ A_h(x, \nabla u_h(x)) &\rightharpoonup B(x, \nabla u(x)) && \text{in } L^2(\Omega)^N, \end{aligned}$$

where u is the unique solution to

$$\begin{aligned} -\nabla \cdot B(x, \nabla u(x)) &= f(x) \quad \text{in } \Omega, \\ u(x) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The following compactness result holds true.

Theorem 11 *Let $\{A_h\}$ be a sequence of operators such that $A_h \in M(\lambda, \mu, \beta, \Omega)$. Then there exists a subsequence that G-converges to some $B \in M\left(\lambda, \frac{\mu^2}{\lambda}, \beta, \Omega\right)$.*

Proof. See [57] and [31]. ■

We now turn to the study of parabolic operators. Svanstedt conducted studies on G-convergence for nonlinear monotone parabolic problems in [55]. There are also some studies on nonlinear monotone parabolic operators that were performed by Pankov in [46]. We investigate

$$\begin{array}{rcll} \partial_t u_h(x,t) - \nabla \cdot A_h(x,t,\nabla u_h) & = & f(x,t) & \text{in } \Omega_T, \\ u_h(x,0) & = & u^0(x) & \text{in } \Omega, \\ u_h(x,t) & = & 0 & \text{on } \partial\Omega \times (0,T), \end{array} \tag{9}$$

where $f \in L^2(0,T;H^{-1}(\Omega))$ and $u^0 \in L^2(\Omega)$. We define $M(c,\lambda,\mu,\beta,\Omega_T)$ to be the set of all functions,

$$A : \Omega_T \times \mathbb{R}^N \to \mathbb{R}^N,$$

which satisfy four structure conditions:

(M_1) $|A(x,t;0)| \leq c$ a.e. in Ω_T;
(M_2) $A(\cdot,\cdot;\xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^N$;
(M_3) $(A(x,t;\xi_1) - A(x,t;\xi_2)) \cdot (\xi_1 - \xi_2) \geq \lambda |\xi_1 - \xi_2|^2$ a.e. in Ω_T for all $\xi_1, \xi_2 \in \mathbb{R}^N$;
(M_4) $|A(x,t;\xi_1) - A(x,\xi_2)| \leq \mu(1 + |\xi|_1 + |\xi_2|)^{1-\beta} |\xi_1 - \xi_2|^\beta$ a.e. in Ω_T for all $\xi_1, \xi_2 \in \mathbb{R}^N$,
where λ, $\mu > 0$ and $0 < \beta \leq 1$.

The next definition contains a G-convergence concept for monotone parabolic operators.

Definition 12 *We say that $\{A_h\} \in M(c,\lambda,\mu,\beta,\Omega_T)$ G-converges to some $B \in M(c',\lambda',\mu',\beta',\Omega_T)$ if, for any $f \in L^2(0,T;H^{-1}(\Omega))$, the solutions u_h to the sequence of problems (9) satisfy*

$$\begin{array}{rcll} u_h(x,t) & \rightharpoonup & u(x,t) & \text{in } L^2(0,T;H_0^1(\Omega)), \\ A_h(x,t,\nabla u_h) & \rightharpoonup & B(x,t,\nabla u) & \text{in } L^2(\Omega_T)^N, \end{array}$$

where u is the unique solution to the problem

$$\begin{array}{rcll} \partial_t u(x,t) - \nabla \cdot B(x,t,\nabla u) & = & f(x,t) & \text{in } \Omega_T, \\ u(x,0) & = & u^0(x) & \text{in } \Omega, \\ u(x,t) & = & 0 & \text{on } \partial\Omega \times (0,T). \end{array}$$

We provide the result as it was proven in [55].

Theorem 13 *Let $\{A_h\}$ be a sequence in $M(c,\lambda,\mu,\beta,\Omega_T)$. Then there exists a subsequence that G-converges to some limit $B \in M(c',\lambda,\widetilde{\mu},\widetilde{\beta},\Omega_T)$, where c' and $\widetilde{\mu}$ are positive constants that depend on c, λ, μ and β. Here, $\widetilde{\beta} = \beta/(2-\beta)$.*

Proof. See Theorem 5.2 in [55]. For more information, see also [56]. ■

Remark 14 *G-convergence results are particularly valuable for evolution problems because they give us the character of the limit problem including the initial conditions if just a few simple assumptions are fullfilled.*

2.2 Two-scale convergence and its generalizations

The two-scale convergence method is very efficient for periodic homogenization and was originally introduced by G. Nguetseng in [44]. Later the concept was further developed by Allaire in [2] as well as by several others in e.g. [42], [7], [43] and [38].

The concept can be generalized to any number of scales and capture rapid oscillations on several levels. In this case, it is called multiscale convergence. It is possible to extend the concept to also include one or several microscopic temporal scales and get evolution multiscale convergence.

Furthermore, we have a similar concept that is called unfolding, which also uses multiple scales but in a different way than in two-scale convergence. See Section 2.3.

2.2.1 Two-scale convergence

The important point about the two-scale limit is that it has a second variable meaning that the limit belongs to another function space compared with the sequence that converges to it. This means that it can capture oscillations on a microscopic level. This type of information is completely lost in the case of usual weak L^2-convergence. Below we state the definition and the compactness result for two-scale convergence in the shape it was presented by Allaire in [2].

Definition 15 *A sequence of functions u_ε in $L^2(\Omega)$ is said to two-scale converge to a limit u_0 belonging to $L^2(\Omega \times Y)$, if we have for any v in $L^2(\Omega; C_\sharp(Y))$, that*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) v\left(x, \frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y u_0(x,y) v\,(x,y)\, dxdy.$$

We write

$$u_\varepsilon(x) \overset{2}{\rightharpoonup} u_0(x,y).$$

If we set certain conditions on the sequence $\{u_\varepsilon\}$ we can guarantee two-scale convergence, at least up to a subsequence. The following compactness theorem is cited from [2].

Theorem 16 *From each bounded sequence $\{u_\varepsilon\}$ in $L^2(\Omega)$ we can extract a subsequence, and there exists a limit $u_0 \in L^2(\Omega \times Y)$, such that this subsequence two-scale converges to u_0.*

Proof. See Theorem 1.2 in [2] and Theorem 1 in [44]. ■

Important results regarding two-scale convergence are provided below.

Theorem 17 *A two-scale limit is unique.*

Proof. See the discussion after Definition 6 in [38]. ■

Theorem 18 *Any function $u_0 \in L^2(\Omega \times Y)$ is attained as a two-scale limit.*

Proof. See Lemma 1.13 in [2]. ■

The two-scale limit has a strong connection with the weak $L^2(\Omega)$-limit. Below we recall this property.

Theorem 19 *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega)$ which two-scale converges to $u_0 \in L^2(\Omega \times Y)$. Then we have*

$$u_\varepsilon(x) \rightharpoonup \int_Y u_0(x,y)dy \text{ in } L^2(\Omega).$$

Proof. See Theorem 6 in [38]. ■

This means that the usual weak limit of the sequence is obtained from the two-scale limit by integrating the last one over Y.

It is also possible to characterize the limits of gradients if we have stronger a priori estimates. We state a theorem, where, in particular, various versions and generalizations of (i) will be useful in the sequel.

Theorem 20 *(i) Let $\{u_\varepsilon\}$ be a bounded sequence in $H^1(\Omega)$ that converges weakly to a limit u in $H^1(\Omega)$. Then,*

$$u_\varepsilon(x) \overset{2}{\rightharpoonup} u(x)$$

and there exists a function u_1 in $L^2(\Omega; H^1_\sharp(Y))$, such that, up to a subsequence,

$$\nabla u_\varepsilon(x) \overset{2}{\rightharpoonup} \nabla u(x) + \nabla_y u_1(x,y).$$

(ii) Let $\{u_\varepsilon\}$ and $\{\varepsilon \nabla u_\varepsilon\}$ be two bounded sequences in $L^2(\Omega)$. Then there exists a function u_0 in $L^2(\Omega; H^1_\sharp(Y))$ such that, up to a subsequence,

$$u_\varepsilon(x) \overset{2}{\rightharpoonup} u_0(x,y)$$

and

$$\varepsilon \nabla u(x) \overset{2}{\rightharpoonup} \nabla_y u_0(x,y).$$

Proof. See Proposition 1.14 in [2]. ■

Below we illustrate a concept of convergence, which we can successfully utilize in some situations where the usual two-scale convergence is not applicable. The concept was first introduced by Holmbom in [33] in an evolution setting, but the name "very weak multiscale convergence" came in use later in [27]; see Definition 34. The concept requires using a smaller class of test functions than in the case of conventional two-scale convergence.

We give the definition of very weak two-scale convergence.

Definition 21 *$\{\varphi_\varepsilon\}$ is said to two-scale converge very weakly to $\varphi_1 \in L^1(\Omega \times Y)$ if*

$$\lim_{\varepsilon \to 0} \int_\Omega \varphi_\varepsilon(x) v_1(x) v_2\left(\frac{x}{\varepsilon}\right) dx = \int_\Omega \int_Y \varphi_1(x,y) v_1(x) v_2(y)\, dydx$$

for all $v_1 \in D(\Omega)$ *and all* $v_2 \in C^\infty_\sharp(Y)/\mathbb{R}$ *and*

$$\int_Y \varphi_1(x,y)dy = 0. \tag{10}$$

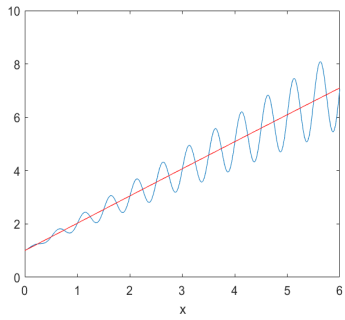
We write

$$\varphi_\varepsilon(x) \overset{2}{\underset{vw}{\rightharpoonup}} \varphi_1(x,y).$$

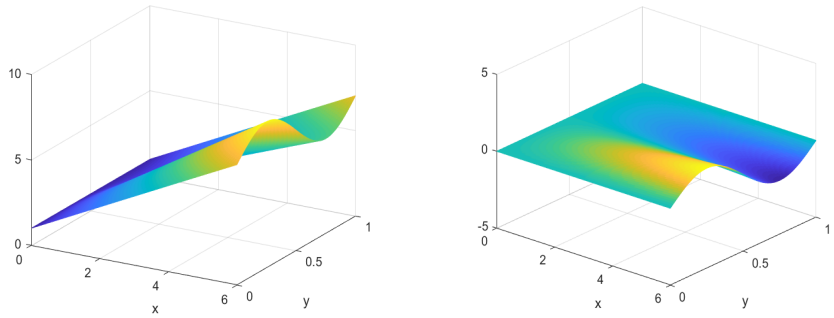
A compactness result that includes very weak two-scale convergence as a special case is found in Theorem 28.

Remark 22 *Due to (10) the very weak two-scale limit is unique.*

We consider the sequence $u_\varepsilon(x) = 1 + x + \frac{1}{20}\sin\left(\frac{x}{4}\right) + \frac{x}{4}\sin\left(\frac{2\pi x}{\varepsilon}\right)$, $x \in \Omega$, where $\Omega = (0,6)$. Here, u_ε is a two-scale oscillating function. As shown in Figure 11a, the weak limit $u(x) = 1 + x + \frac{1}{20}\sin\left(\frac{x}{4}\right)$ captures the global trend, but it misses the microscopic oscillations of u_ε. To capture these oscillations we use two-scale convergence. This sequence two-scale converges to $u_0(x,y) = 1 + x + \frac{1}{20}\sin\left(\frac{x}{4}\right) + \frac{x}{4}\sin(2\pi y)$, as $\varepsilon \to 0$. Figure 11b shows that the two-scale limit follows the weak limit in the x-direction and captures the microscopic oscillations in the y-direction. Figure 11c shows the very weak two-scale limit. In our case, the very weak two-scale limit is $g_0(x,y) = \frac{x}{4}\sin(2\pi y)$. Here, the global trend has been lost, but the information about the micro-oscillations remains. This particular sequence $\{u_\varepsilon\}$ is bounded in $L^2(\Omega)$ and hence possesses all three types of limits.



a) u_ε *for* $\varepsilon = 2^{-1}$ *and its weak limit.*



b) The two-scale limit of $\{u_\varepsilon\}$. *c) The very weak limit of* $\{u_\varepsilon\}$.

Figure 11. *The sequence* $\{u_\varepsilon\}$, *for a fixed* ε, *together with the weak limit, two-scale limit and very weak two-scale limit.*

2.2.2 Multiscale convergence

Multiscale convergence is a generalization of two-scale convergence that allows more scales and, in turn, captures more types of microscopic oscillations. The concept of multiscale convergence was introduced in [7] for the homogenization of partial differential equations with oscillating coefficients with an arbitrary number of scales. We denote $y^n = (y_1, \dots, y_n)$, $dy^n = dy_1 \dots dy_n$, $Y^n = Y_1 \times \dots \times Y_n$, where $Y_k = Y = (0,1)^N$ and let $\varepsilon_k(\varepsilon)$, $k = 1, \dots, n$ be given strictly positive functions such that $\varepsilon_k(\varepsilon) \to 0$ when $\varepsilon \to 0$.

Definition 23 *A sequence* $\{u_\varepsilon\}$ *in* $L^2(\Omega)$ *is said to* $(n+1)$*-scale converge to* $u_0 \in L^2(\Omega \times Y^n)$ *if*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) v\left(x, \frac{x}{\varepsilon_1}, \cdots, \frac{x}{\varepsilon_n}\right) dx = \int_\Omega \int_{Y^n} u_0(x, y^n) v(x, y^n) dy^n dx$$

for any $v \in L^2(\Omega; C_\sharp(Y^n))$. *We denote this convergence by*

$$u_\varepsilon(x) \overset{n+1}{\rightharpoonup} u_0(x, y^n).$$

To get a better understanding of the relation between the scales and obtain compactness results we need some assumptions. Next, we give the definition of separated and well-separated scales, respectively, from paper [7].

Definition 24 *If*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0$$

we say that the scales are separated. When

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon_k} \left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^m = 0$$

for a positive integer m *the scales are called well-separated.*

Below we state a compactness theorem.

Theorem 25 *If $\{u_\varepsilon\}$ is bounded in $L^2(\Omega)$ and the scales ε_k, $k = 1, \ldots, n$ are separated then there exists a function $u_0 \in L^2(\Omega \times Y^n)$ such that*

$$u_\varepsilon(x) \overset{n+1}{\rightharpoonup} u_0(x, y^n),$$

up to a subsequence.

Proof. See the proof of Theorem 2.4 in [7]. ■

We give the following compactness result for sequences of gradients.

Theorem 26 *Let $\{u_\varepsilon\}$ be a bounded sequence in $H^1(\Omega)$. If the scales are separated, there exists a function $u \in H^1(\Omega)$ and n functions $u_1 \in L^2(\Omega; H^1_\sharp(Y_1))$ and $u_k \in L^2(\Omega \times Y^{k-1}; H^1_\sharp(Y_k))$, $k = 2, \ldots, n$, such that, up to a subsequence,*

$$u_\varepsilon(x) \overset{n+1}{\rightharpoonup} u(x),$$

and

$$\nabla u_\varepsilon(x) \overset{n+1}{\rightharpoonup} \nabla u(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y^k).$$

Proof. See the proof of Theorem 2.6 in [7]. ■

The following definition of very weak multiscale convergence is found in [27].

Definition 27 *We say that $\{g_\varepsilon\}$ $(n+1)$-scale converges very weakly to $g_0 \in (\Omega \times Y^n)$ if*

$$\int_\Omega g_\varepsilon(x) v_1\left(x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \cdots, \frac{x}{\varepsilon_{n-1}}\right) v_2\left(\frac{x}{\varepsilon_n}\right) dx$$
$$\to \int_\Omega \int_{Y^n} g_0(x, y^n) v_1(x, y^{n-1}) v_2(y_n) dy^n dx,$$

for any $v_1 \in D\left(\Omega; C^\infty_\sharp(Y^{n-1})\right)$ and $v_2 \in C^\infty_\sharp(Y_n)/\mathbb{R}$, where

$$\int_{Y_n} g_0(x, y^n) dy_n = 0.$$

We write

$$g_\varepsilon(x) \overset{n+1}{\underset{vw}{\rightharpoonup}} g_0(x, y^n).$$

An important application of this kind of convergence is its ability to handle certain sequences of the type $\left\{\varepsilon^{-1} u_\varepsilon\right\}$. Of course, these sequences are not necessarily bounded in $L^2(\Omega)$. This was originally investigated for sequences of the type $\left\{\varepsilon^{-1}(u_\varepsilon - u)\right\}$ in an evolution setting, where u is the weak limit of $\{u_\varepsilon\}$ in $L^2(0, T; H^1_0(\Omega))$; see [33]. The compactness result for very weak multiscale convergence is the following:

Theorem 28 *Let $\{u_\varepsilon\}$ be a bounded sequence in $H_0^1(\Omega)$ and assume that the scales are well-separated. Then there exists a subsequence such that*

$$\frac{u_\varepsilon(x)}{\varepsilon_n} \overset{n+1}{\underset{vw}{\searrow}} u_n(x, y^n),$$

where $u_1 \in L^2(\Omega; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_n \in L^2(\Omega \times Y^{n-1}; H^1_\sharp(Y_n)/\mathbb{R})$ for $n = 2, 3, \ldots$ are the same as in Theorem 26.

Proof. See Theorem 4 in [27]. ■

Theorem 29 *Very weak multiscale limits are unique.*

Proof. See Remark 2.52 in [48]. ■

This kind of convergence can also be extended to include sequences of time-dependent functions, which will be studied in subsection 2.2.3 below. More informative texts on very weak multiscale convergence and related concepts can be found in e.g. [33], [45] and [48]. Moreover, the concept is beneficial for the detection of scales; see [28].

2.2.3 Evolution multiscale convergence

This subsection is devoted to the study of convergences of sequences that have oscillations in both time and space in terms of multiscale convergence. When we add scales with fast microoscillations in time everything relating to multiscale convergence needs to be adapted. We let $S = (0, 1)$ and $S^m = S_1 \times \ldots \times S_m$, $S_j = S$ for $j = 1, \ldots, m$. We denote $\mathcal{Y}_{n,m} = Y^n \times S^m$, $ds^m = ds_m \ldots ds_1$ and $s^m = s_1, \ldots, s_m$. Moreover, we let the scales $\varepsilon_k(\varepsilon)$, $k = 1, \ldots, n$ and $\varepsilon'_j(\varepsilon)$, $j = 1, \ldots, m$, be given strictly positive functions that goes to zero when ε does. Using the introduced notations we state the definition of evolution multiscale convergence.

Definition 30 *The sequence $\{u_\varepsilon\}$ is said to $(n+1, m+1)$-scale converge to $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ if*

$$\begin{aligned} &\lim_{\varepsilon \to 0} \int_{\Omega_T} u_\varepsilon(x,t) v\left(x, t, \frac{x}{\varepsilon_1}, \cdots, \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon'_1}, \cdots, \frac{t}{\varepsilon'_m}\right) dxdt \\ = &\int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} u_0(x, t, y^n, s^m) v(x, t, y^n, s^m) dy^n ds^m dxdt, \end{aligned}$$

for all $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{n,m}))$. We write

$$u_\varepsilon(x,t) \overset{n+1,m+1}{\searrow} u_0(x, t, y^n, s^m).$$

To come closer to some main results of evolution multiscale convergence we give the definition of jointly separated and jointly well-separated lists of scales that were originally introduced by Persson in [47].

Definition 31 *Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ be lists of (well-)separated scales. Collect all elements from both lists in one common list. If, out of the possible duplicates, i.e. scales which tend to zero equally fast as ε tends to zero, one member of each pair is removed and the list in order of magnitude of all the remaining elements is (well-)separated, the lists $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ are said to be jointly (well-)separated.*

The following is a compactness result for $(n+1, m+1)$-scale convergence.

Theorem 32 *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(\Omega_T)$ and assume that the lists $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ are jointly separated. Then there exists a function $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m})$ such that*

$$u_\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x,t,y^n,s^m),$$

up to subsequence.

Proof. See Theorem 17 in [30] and also Theorem 2.66 in [48]. ■

In the following theorem, we will state an $(n+1, m+1)$-scale convergence result for sequences of gradients of time-dependent functions. The space $W_2^1\left(0,T; H_0^1(\Omega), L^2(\Omega)\right)$ in the theorem below is the space of all functions in $L^2(0,T; H_0^1(\Omega))$ such that the time derivative belongs to $L^2(0,T; H^{-1}(\Omega))$.

Theorem 33 *Let $\{u_\varepsilon\}$ be a bounded sequence in $W_2^1\left(0,T; H_0^1(\Omega), L^2(\Omega)\right)$ and assume that the lists $\{\varepsilon_1, \ldots, \varepsilon_n\}$ and $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ are jointly well-separated. Then there exists a subsequence such that*

$$u_\varepsilon(x,t) \to u(x,t) \text{ in } L^2(\Omega_T),$$

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \text{ in } L^2(0,T; H_0^1(\Omega))$$

and

$$\nabla u_\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x,t,y^k,s^m),$$

where $u \in W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$, $u_1 \in L^2(\Omega_T \times S^m; H_\sharp^1(Y_1)/\mathbb{R})$ and $u_k \in L^2(\Omega_T \times \mathcal{Y}_{k-1,m}; H_\sharp^1(Y_k)/\mathbb{R})$ for $k = 2, \ldots, n$.

Proof. See Theorem 2.74 in [48]. ■

Next, we define very weak evolution multiscale convergence. See e.g. [48] or [30].

Definition 34 *We say that the sequence $\{g_\varepsilon\}$ $(n+1, m+1)$-scale converges very weakly to $g_0 \in L^1(\Omega_T \times \mathcal{Y}_{n,m})$ if*

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} g_\varepsilon(x,t) v_1\left(x, \frac{x}{\varepsilon_1}, \cdots, \frac{x}{\varepsilon_{n-1}}\right) v_2\left(\frac{x}{\varepsilon_n}\right) c\left(t, \frac{t}{\varepsilon'_1}, \cdots, \frac{t}{\varepsilon'_m}\right) dxdt$$
$$= \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} g_0(x,t,y^n,s^m) v_1\left(x, y^{n-1}\right) v_2(y_n) c(t,s^m) dy^n ds^m dxdt,$$

for all $v_1 \in D(\Omega; C^\infty_\sharp(Y^{n-1}))$, $v_2 \in C^\infty_\sharp(Y_n)/\mathbb{R}$ *and* $c \in D(0,T; C^\infty_\sharp(S^m))$, *where*

$$\int_{Y_n} g_0(x,t,y^n,s^m)dy_n = 0.$$

We write

$$g_\varepsilon(x,t) \overset{n+1,m+1}{\underset{vw}{\rightharpoonup}} g_0(x,t,y^n,s^m).$$

A compactness result for very weak evolution multiscale convergence follows below.

Theorem 35 *Let* $\{u_\varepsilon\}$ *be a bounded sequence in* $W^1_2(0,T; H^1_0(\Omega), L^2(\Omega))$ *and assume that the lists* $\{\varepsilon_1, \ldots, \varepsilon_n\}$ *and* $\{\varepsilon'_1, \ldots, \varepsilon'_m\}$ *are jointly well-separated. Then, up to a subsequence,*

$$\frac{u_\varepsilon(x,t)}{\varepsilon_n} \overset{n+1,m+1}{\underset{vw}{\rightharpoonup}} u_n(x,t,y^n,s^m),$$

where $u_1 \in L^2(\Omega_T \times S^m;\ H^1_\sharp(Y_1)/\mathbb{R})$ *and* $u_n \in L^2(\Omega_T \times \mathcal{Y}_{n-1,m};\ H^1_\sharp(Y_n)/\mathbb{R})$ *for* $n = 2, 3 \ldots$ *are the same as in Theorem 33.*

Proof. See Theorem 2.78 in [48] or Theorem 7 in [30]. ∎

2.3 Classical ingredients of the periodic unfolding method

An alternative approach to two-scale convergence is the periodic unfolding method. The initial idea of the method belongs to Arbogast, Douglas and Hornung in [9], where "dilation operator" is used for the analysis of a periodically perforated medium. Cioranescu, Damlamian and Griso introduced the notion of the unfolding method in its present form and gave it the name unfolding, [12]. Further development of the concept can be found in e.g. [13], [14], [15]. In this subsection, we'll define the notion of unfolding and formulate some of the main results.

We assume that $\Omega \subset \mathbb{R}^N$ is an open bounded set with a smooth boundary. Moreover, in the figures below we chose $Y = (0,1)^N$ and let $\{Y_i\}$ be a paving of $\mathbb{R}^N$ by disjoint open unit cubes such that $\bigcup_{i=1}^\infty \overline{Y_i}$ covers $\mathbb{R}^N$. We define $[z]_Y$ for every $z \in \mathbb{R}^N$ as the "beginning" of the cell Y_i, where z is located. This corresponds to the bottom left corner in two dimensions. See figure 12 below. We chose a paving with unit cubes for the sake of simplicity.

Remark 36 *Many other shapes of* Y_i *are allowed and hence* $|Y|$ *appears in some results even though* $|Y| = 1$ *for a unit cube. See Section 2.1 in [13]. Many of the properties of the unfolding operator also apply under less restrictive assumptions on* Ω.

We set

$$\{z\}_Y = z - [z]_Y.$$

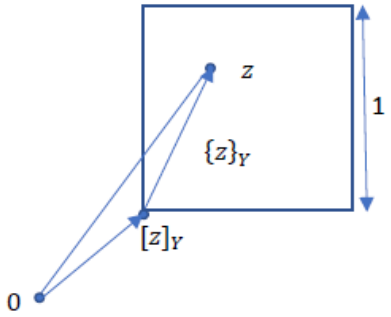


Figure 12. *The unit cell* Y_i.

According to Figure 12, $\{z\}_Y$ describes the position of z in the current unit cell Y_i. By scaling by $1/\varepsilon$ we get (see Figure 13)

$$x = \varepsilon \left(\left[\frac{x}{\varepsilon} \right]_Y + \left\{ \frac{x}{\varepsilon} \right\}_Y \right).$$

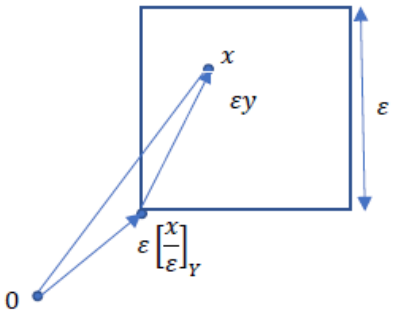


Figure 13. *Two-scale decomposition of* $x \in \Omega$.

The set $\widehat{\Omega}_\varepsilon$ contains the interior of the union of cells $\overline{Y_i^\varepsilon}$, which are located entirely inside Ω and $\Lambda_\varepsilon = \Omega - \widehat{\Omega}_\varepsilon$; see Figure 14. See also Section 2.1 in [13].

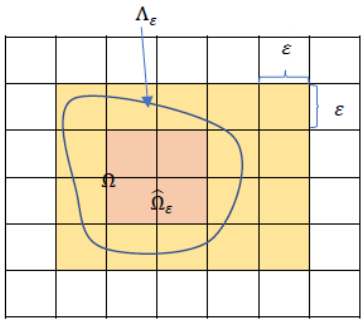


Figure 14. *The set* Ω *covered by* ε*-cells.*

Below we define the periodic unfolding operator τ_ε given by Cioranescu et al. in [13].

Definition 37 *For u Lebesgue-measurable on Ω, the unfolding operator τ_ε is defined as follows:*

$$\tau_\varepsilon(u)(x,y) = \begin{cases} u\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & a.e. \quad on \quad \widehat{\Omega}_\varepsilon \times Y, \\ 0 & a.e. \quad on \quad \Lambda_\varepsilon \times Y. \end{cases}$$

Next we present some properties of the unfolding operator τ_ε. See Proposition 2.5 and the formulas (2.2) and (3.1) in [13].

Proposition 38 *Let τ_ε be the unfolding operator. For $\varepsilon > 0$, we have*
(i) $\tau_\varepsilon(vw) = \tau_\varepsilon(v)\tau_\varepsilon(w)$ for v and w Lebesgue measurable.
(ii) $\tau_\varepsilon : L^p(\Omega) \to L^p(\Omega \times Y)$. τ_ε is a continuous and linear operator for $p \in [1, +\infty)$.
(iii) For all $u \in L^1(\Omega)$

$$\frac{1}{|Y|}\int_\Omega \int_Y |\tau_\varepsilon(u)(x,y)|\, dydx \leq \int_\Omega |u(x)|\, dx,$$

holds.
(iv) For every $u \in L^p(\Omega)$, $p \in [1, +\infty)$, one has

$$\|\tau_\varepsilon(u)\|_{L^p(\Omega\times Y)} = |Y|^{\frac{1}{p}} \left\| u\chi_{\widehat{\Omega}_\varepsilon} \right\|_{L^p(\Omega)} \leq |Y|^{\frac{1}{p}} \|u\|_{L^p(\Omega)}.$$

(v) $\nabla_y \tau_\varepsilon(u)(x,y) = \varepsilon\tau_\varepsilon(\nabla u)(x,y)$ for every $u \in W^{1,p}(\Omega)$.

A detailed discussion of the unfolding operator´s relation to two-scale convergence can be found in Paper VI. In the context of two-scale convergence we give the following proposition from [13].

Proposition 39 *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$ with $p \in (1, +\infty)$. The following assertions are equivalent:*

(i) $\{\tau_\varepsilon(u_\varepsilon)\}$ converges weakly to u_0 in $L^p(\Omega \times Y)$,
(ii) $\{u_\varepsilon\}$ two-scale converges to u_0.

Proof. See Proposition 2.14 in [13]. ■

Next, we present the averaging operator $\mathcal{U}_\varepsilon$ that is the adjoint of the unfolding operator τ_ε. The definition below is found in [13].

Definition 40 *The averaging operator $\mathcal{U}_\varepsilon : L^p(\Omega \times Y) \to L^p(\Omega)$, $p \in [1, +\infty]$ defined as*

$$\mathcal{U}_\varepsilon(v)(x) = \begin{cases} \frac{1}{|Y|}\int_Y v\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right) dz & a.e. \ on \ \widehat{\Omega}_\varepsilon, \\ 0 & a.e. \ on \ \Lambda_\varepsilon. \end{cases}$$

Remark 41 *In this case* $\left\{\frac{x}{\varepsilon}\right\}_Y$ *freezes the* Y*-variable to one position in the* Y*-cell at a time. In turn,* $\varepsilon\left[\frac{x}{\varepsilon}\right]_Y$ *selects the one* ε*-cell, where* x *is located and* εz *takes us around this cell as* z *passes through the unit cell* Y*. Added to everything else, the operator can be seen as a special case of a generalization of two-scale convergence discussed in Paper VI, where the usual two-scale convergence occurs as another special case, with less restrictive classes of allowed test functions for* $\{\mathcal{U}_\varepsilon\}$*. See e.g. [51], [32] and [34].*

As a consequence, we have

$$\frac{1}{|Y|}\int_\Omega\int_Y \tau_\varepsilon(u)(x,y)v(x,y)dydx = \int_\Omega u(x)\mathcal{U}_\varepsilon(v)(x)dx,$$

for $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega \times Y)$, $p \in [1,\infty]$. For Y the unit cube, we of course have $|Y| = 1$.

The following result can be found as Proposition 2.17 in [13].

Proposition 42 *The averaging operator is linear and continuous from* $L^p(\Omega \times Y)$ *to* $L^p(\Omega)$*,* $p \in [1,+\infty]$*. It holds that*

$$\|\mathcal{U}_\varepsilon(v)\|_{L^p(\Omega)} \le |Y|^{-1/p}\,\|v\|_{L^p(\Omega\times Y)}\,.$$

An example. We let $\Omega = (0,4)$, $Y = (0,1)$ and

$$u_\varepsilon(x) = \frac{2}{5}\sin\left(\frac{2\pi x}{\varepsilon}\right) + x.$$

Then

$$(\tau_\varepsilon u_\varepsilon)(x,y) = \frac{2}{5}\sin(2\pi y) + \left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right).$$

Figure 15 below illustrates how $\tau_\varepsilon u_\varepsilon$ approaches a limit when $\varepsilon \to 0$. It turns out that the limit is

$$u_0(x,y) = \frac{2}{5}\sin(2\pi y) + x$$

in the sense of Proposition 39.

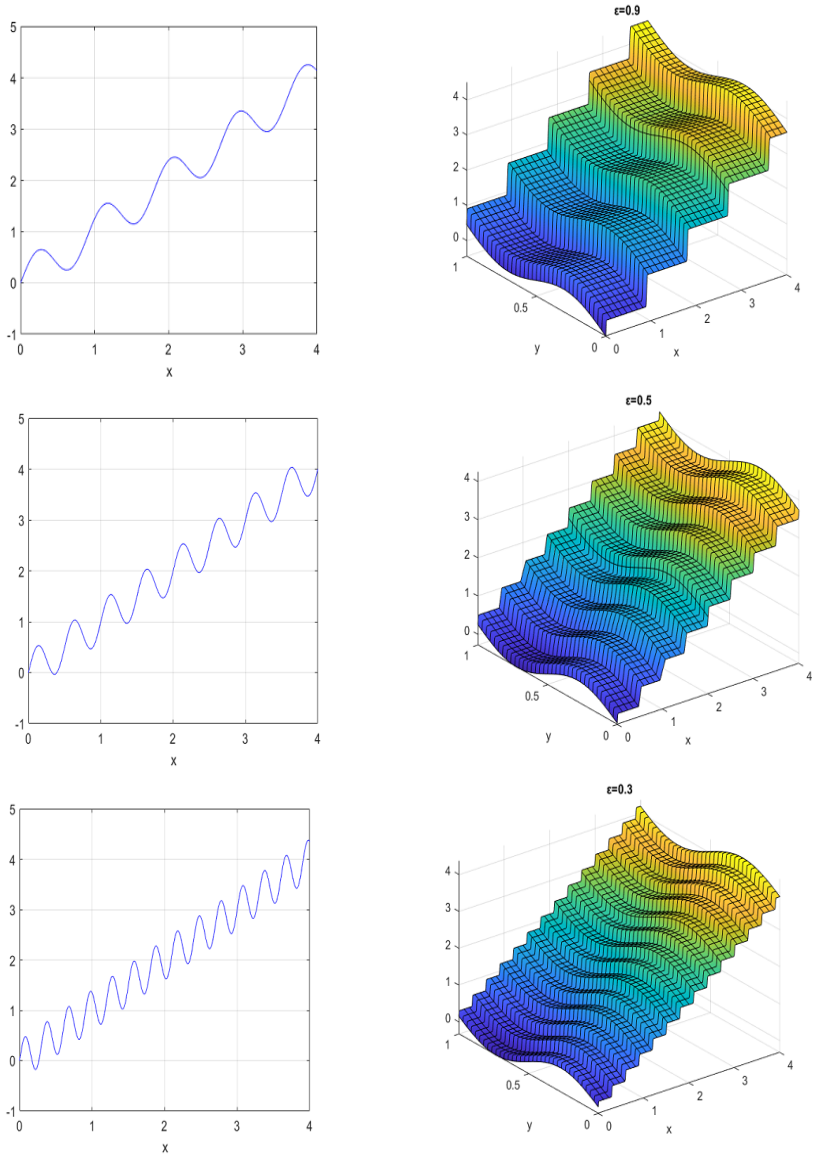


Figure 15. *Functions* $u_\varepsilon(x) = \dfrac{2}{5}\sin\left(\dfrac{2\pi x}{\varepsilon}\right) + x$ *and* $(\tau_\varepsilon u_\varepsilon)(x,y)$ *for* $\varepsilon = 0.9,\ 0.5,\ 0.3$.

3 Homogenization in perforated domains

The mathematical approach of homogenization theory can also be used for boundary-value problems defined on perforated domains. The complicated structure of the perforated domains makes it difficult to perform numerical calculations. These types of formations are very important for engineering applications such as heat exchangers in the energy sector, foams and oil exploitations.

To illustrate the geometrical structure of a perforated domain, we first introduce Ω as an open bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. By a perforated domain Ω_ε we mean a domain where periodically arranged identical holes, with a period of characteristic size ε, namely Y_ε^H (dark area in the cube, Figure 16), have been removed from Ω, except in the layer closest to the boundary $\partial\Omega$. For more details, see e.g. [37]. We let Ω and Ω_ε belong to $\mathbb{R}^N$, $N = 2, 3$ throughout Chapter 3.

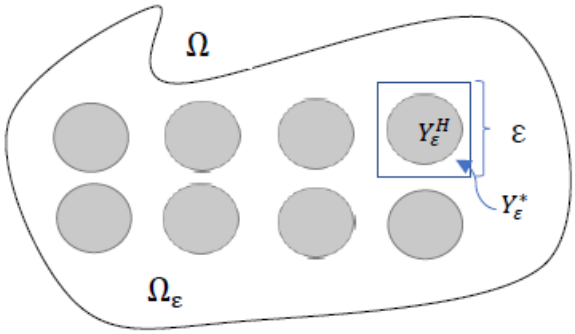


Figure 16. *A two-dimensional case for the region* Ω_ε.

3.1 Mixed problems in perforated domains

Homogenization problems defined on perforated domains are based on different kinds of extension operators, which make it possible to embed a function space of extended functions into the initial one in perforated media in a sense that preserves boundedness. The smoothness and regularity of the domain are essential qualities for these types of embeddings. This section is devoted to studying a linear elliptic problem with Neumann data on the boundary of the perforations and Dirichlet data on the exterior boundary, where A is Y-periodic:

$$\begin{aligned} -\nabla \cdot \left(A\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\right) &= f_\varepsilon(x) && \text{in } \Omega_\varepsilon, \\ A\left(\tfrac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\cdot n &= 0 && \text{on } \partial\Omega_\varepsilon - \partial\Omega, \\ u_\varepsilon(x) &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{11}$$

where f_ε is the restriction of some function $f \in L^2(\Omega)$ to Ω_ε.

The variational formulation of the mixed problem (11) is to find $u_\varepsilon \in H^1(\Omega_\varepsilon)$, which satisfies the boundary conditions in (11), such that

$$\int_{\Omega_\varepsilon} A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon(x)\cdot\nabla v(x)dx = \int_{\Omega_\varepsilon} f_\varepsilon(x)v(x)dx,$$

for all $v \in H^1(\Omega_\varepsilon)$ with $v = 0$ on $\partial\Omega$.

The a priori estimate

$$\|u_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leq C \tag{12}$$

follows from the variational formulation of (11). See e.g. [15]. The primary obstacle is that all solutions u_ε are located in different function spaces. To be able to use the weak compactness of reflexive Banach spaces the entire sequence has to be in the same function space. When using the Neumann data on the boundary of the holes the classical technique is to find extensions $\widetilde{u}_\varepsilon$ of the solutions from the solid part Ω_ε to the whole material Ω, maintaining the a priori estimate (12), such that

$$\|\widetilde{u}_\varepsilon\|_{H^1(\Omega)} \leq C,$$

where C is independent of ε, but does not necessarily have to be the same as in (12). See e.g. [1] and [10].

An extension from $L^2(\Omega_\varepsilon)$ to $L^2(\Omega)$ can be done easily. If $u_\varepsilon \in L^2(\Omega_\varepsilon)$ and $\widetilde{u}_\varepsilon$ is an extension by zero, it will belong to $L^2(\Omega)$, and if the sequence $\{u_\varepsilon\}$ is bounded in $L^2(\Omega_\varepsilon)$ then $\{\widetilde{u}_\varepsilon\}$ will be bounded in $L^2(\Omega)$. An extension from $H^1(\Omega_\varepsilon)$ to $H^1(\Omega)$ is not that simple since the boundedness of the gradient has to be preserved.

The homogenized problem is elliptic and reads as

$$\begin{aligned} -\nabla \cdot (B\nabla u(x)) &= \phi f(x) && \text{in } \Omega, \\ u(x) &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where u is defined as a certain kind of limit for $\{\widetilde{u}_\varepsilon\}$ and ϕ is the material volume fraction. The separation of variables via

$$u_1(x, y) = \nabla u(x) \cdot z(y)$$

get us the so-called local problems

$$\begin{aligned} -\nabla_y \cdot (A(y)\,(e_i + \nabla_y z_i(y))) &= 0 && \text{in } Y^*, \\ A(y)\,(e_i + \nabla_y z_i(y)) \cdot n &= 0 && \text{on } \partial Y^* - \partial Y, \end{aligned}$$

$i = 1, \ldots, N$. Out of this we can compute the following homogenized coefficient

$$B_{ij} = \int_{Y^*} (A_{ij}(y) + A(y)\nabla_y z_i(y)) \cdot e_j dy,$$

where e_i is the orthonormal basis of $\mathbb{R}^N$ and z_i are unique solutions of the cell problems. Note that only the material part Y^* is used to compute B and hence the extension to all of Ω is only a technical step to allow a passage to the limit.

Remark 43 *The existence of an extension with a preserved a priori estimate with strong restrictions on Ω_ε was first provided by Cioranescu and Saint Jean Paulin in [10], where the authors studied the elastic torsion of a cylindrical bar. [1] contains more advanced extension techniques that work for a significantly*

larger class of perforated domains. In [2], Allaire introduced a new method based on two-scale convergence, which only uses extensions by zero and thus only the L^2 estimate is preserved. The point is that both the solution u_ε and its gradient ∇u_ε are extended by zero and the rest of the results are obtained by two-scale convergence techniques. We expand these results to fit certain evolution problems in Paper I. See also e.g. [8].

We also refer the reader to [7], where the results in [2] were broadened to the multiscale case. The case of inhomogeneous Neumann data on the boundary of the perforations is treated for linear elliptic problems with one rapid scale with unfolding methods in [15].

3.2 Dirichlet problems in perforated domains

In this section, we outline first the Poisson equation in a periodically perforated domain with a Dirichlet condition on the boundaries of the perforations. The most interesting part here is when the hole size goes to zero faster than the period of the microstructure. We then illustrate how the Stokes equations in a porous medium give Darcy's law through a homogenization process.

3.2.1 The classical strange term

This subsection is devoted to a characterization of the so-called "strange term" when the perforations have the form of balls of some critical size periodically distributed where we apply homogeneous boundary condition both on the boundary of Ω and the boundary of the hole. See e.g. [16], [18], [19] and [23].

Let us first consider the Dirichlet problem of the type

$$\begin{array}{rcccl} -\triangle u_\varepsilon(x) & = & f(x) & \text{in} & \Omega_\varepsilon, \\ u_\varepsilon(x) & = & 0 & \text{on} & \partial\Omega_\varepsilon, \end{array} \tag{13}$$

where $\Omega_\varepsilon \subset \Omega \subset \mathbb{R}^N$, $N = 2, 3$ is an open bounded set perforated by periodically arranged holes (see Figure 16) whose shape and fraction of Ω is kept constant for different values of ε and $f \in L^2(\Omega)$.

The variational formulation means that we search for $u_\varepsilon \in H_0^1(\Omega_\varepsilon)$ such that

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon(x) \cdot \nabla v(x) dx = \int_{\Omega_\varepsilon} f(x) v(x) dx, \tag{14}$$

for all $v \in H_0^1(\Omega_\varepsilon)$.

To obtain the a priori estimate, we'll cite a type of Poincaré inequality for $H_0^1(\Omega_\varepsilon)$ which can be found in Lemma 1 in the Appendix of [59].

Lemma 44 *The constant of the Friedrichs inequality in Ω_ε is of the form $C\varepsilon^2$, i.e.*

$$\int_{\Omega_\varepsilon} |u(x)|^2 \, dx \leq C\varepsilon^2 \int_{\Omega_\varepsilon} |\nabla u(x)|^2 \, dx.$$

Letting $v = u_\varepsilon$ in (14) and using Lemma 44 we get the following estimate

$$\int_{\Omega_\varepsilon} \nabla u_\varepsilon(x) \cdot \nabla u_\varepsilon(x) dx = \int_{\Omega_\varepsilon} f(x) u_\varepsilon(x) dx$$
$$\leq \|f\|_{L^2(\Omega_\varepsilon)} \|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C \|f\|_{L^2(\Omega_\varepsilon)} \varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)^N},$$

which gives us

$$\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)^N} \leq C' \varepsilon \|f\|_{L^2(\Omega_\varepsilon)}.$$

After some root extraction and by using Lemma 44 one more time we get

$$\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)^N} \leq C'' \varepsilon^2,$$

where C'' is a positive constant which does depend on f but not on ε. This means that $\left\{\varepsilon^{-2} u_\varepsilon\right\}$ is bounded in $L^2(\Omega_\varepsilon)$ and even in $L^2(\Omega)$ after extension by zero from Ω_ε to Ω. For this reason, it will be possible to find a subsequence of $\left\{\varepsilon^{-2} \widetilde{u}_\varepsilon\right\}$ that converges weakly in $L^2(\Omega)$, where $\widetilde{u}_\varepsilon$ is the extension by zero of u_ε to Ω.

Based on the equation (13) above we discuss a certain kind of phenomenon that can arise in the limit. This was first studied by Cioranescu and Murat in [16]. Below we will follow the more recent approach of Dupuy et. al in [23]. We let the holes be spheres centered at $2\pi\varepsilon\mathbb{Z}^N$ such that their radiuses tend to zero faster than the period of the microstructure. This means that

$$\lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon} = 0,$$

where by r_ε we mean the radius of the holes and $r_\varepsilon = r(\varepsilon)$, $r_\varepsilon < \pi\varepsilon$, is a sequence of positive numbers tending to zero as ε does. In [16], the authors showed the appearance of a so-called strange term, which appears when the hole radius is equal to a critical size. Obviously, we can see a quite different asymptotic behavior of u_ε, depending on how the hole's radius goes to zero.

This means that we have different characterizations of the limit u depending on the radius of the hole. There are three different kinds of limits. Note that R below is a real function, depending on the dimension of the space, which is given by the following formula; see [23]:

$$R(x) = \begin{cases} -\frac{1}{\ln x} & \text{if } N = 2, \\ x & \text{if } N = 3. \end{cases}$$

We divide the main homogenization result into three cases:

Case 1. *Small holes*

If

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} R\left(\frac{r_\varepsilon}{\varepsilon}\right) = 0,$$

then $\widetilde{u}_\varepsilon \rightharpoonup u$ in $H_0^1(\Omega)$, where u solves

$$\begin{aligned} -\triangle u(x) &= f(x) \quad \text{in} \quad \Omega \\ u(x) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

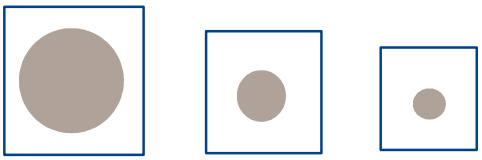


Figure 17. *Small holes.*

The holes shrink and become such a small part of Ω that it cannot be noticed in the limit that holes once existed.

Case 2. *Large holes*

If

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} R\left(\frac{r_\varepsilon}{\varepsilon}\right) = +\infty,$$

then u is identically equal to zero.

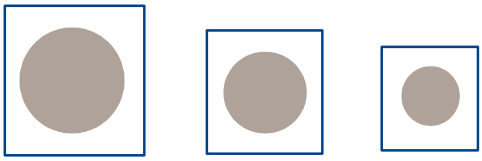


Figure 18. *Large holes.*

The holes are so large that the homogeneous Dirichlet boundary conditions on their surface pull down u_ε to zero in the limit. [4] contains an illustrative discussion on Poincare's inequality for perforations with different types of asymptotic behavior.

Case 3. *Critical size*

If

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} R\left(\frac{r_\varepsilon}{\varepsilon}\right) = l,\ l > 0,$$

then the limit u is the unique solution to the modified Poisson equation

$$\begin{aligned} -\triangle u(x) + (\mu l)\, u(x) &= f(x) \quad \text{in} \quad \Omega, \\ u(x) &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where μ is a positive constant which is defined as

$$\mu = S_N \varphi_1^2(0),$$

where φ_1 is the first normalized eigenvector of the periodic Laplace operator in $Y = [-\pi, \pi)^N$ and S_N is the surface of the sphere in $\mathbb{R}^N$. The most interesting case here is the third case, when all holes happen to have an exceptional critical size. The limit u is the solution to a Dirichlet problem which is the sum of the original Laplace operator and a strange term that comes from nowhere, even though the holes fraction of the domain goes to zero, unlike in Case 1, where the

limit equation is untouched by the holes. The limit problem has been affected by the occurrence of holes, despite their fraction of Ω shrinking to zero during the limit process.

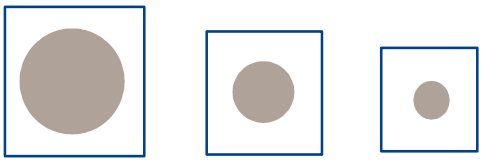


Figure 19. *Critical size.*

3.2.2 A Stokes problem in a porous medium

This subsection deals with the steady Stokes equation in periodically perforated domains with homogeneous Dirichlet boundary conditions. The idea is found in e.g. the articles [59] and [3]. The Stokes equation reads

$$\begin{aligned} \nabla p_\varepsilon(x) - \triangle u_\varepsilon(x) &= f(x) \text{ in } \Omega_\varepsilon, \\ \nabla \cdot u_\varepsilon(x) &= 0 \text{ in } \Omega_\varepsilon, \\ u_\varepsilon(x) &= 0 \text{ on } \partial\Omega_\varepsilon, \end{aligned} \tag{15}$$

where p_ε is the pressure and u_ε the velocity vector of the fluid. Ω_ε is as in Figure 16, where the volume fraction of the holes is the same regardless of ε. This is in line with Case 2 in subsection 3.2.1. Here, vector f is the density of forces. In Figure 20 we illustrate the geometry of a unit cell in porous media.

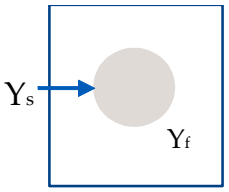


Figure 20. *Geometry of a unit cell in a porous medium.*

Note that u_ε slows down to zero when ε goes to zero and we must study $\varepsilon^{-2}u_\varepsilon$ to obtain a meaningful limit. The weak formulation of (15) has a unique solution $u_\varepsilon \in H_0^1(\Omega_\varepsilon)^N$ and $p_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}$. To be able to perform the homogenization procedure we have to solve the following cell problem:

$$\begin{aligned} \nabla q_k(y) - \triangle v_k(y) &= e_k \quad \text{in } Y, \\ v_k(y) &= 0 \quad \text{on } \partial Y_S, \\ \nabla \cdot v_k(y) &= 0 \quad \text{in } Y, \text{ for } k = 1, 2, \ldots, N. \end{aligned}$$

Here, v_k, q_k are Y-periodic. We compute the permeability tensor

$$\widehat{K}_{ij} = \frac{1}{|Y|} \int_Y \left(v_i^j(y) \right) dy$$

and arrive at Darcy's law

$$\begin{aligned} \nabla \cdot u(x) &= 0 \text{ in } \Omega, \\ u(x) \cdot n &= 0 \text{ on } \partial\Omega, \\ u(x) &= \widehat{K}\left(f(x) - \nabla p(x)\right) \text{ in } \Omega, \end{aligned}$$

where u is the limit obtained in Theorem 46 below.

Remark 45 *We define extensions $\widetilde{u}_\varepsilon$ of u_ε and $\widetilde{p}_\varepsilon$ of p_ε to Ω. The main result is shown in the following theorem. u_ε is extended by zero into the perforations while the extension of p_ε is more complicated. See [3]. Note that here $p_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R}$ means p_ε is unique up to an additive constant.*

Theorem 46 *Let u_ε and p_ε be defined by (15). As $\varepsilon \to 0$, then*

$$\begin{aligned} \frac{\widetilde{u}_\varepsilon(x)}{\varepsilon^2} &\rightharpoonup u(x) \quad \textit{weakly in } L^2(\Omega), \\ \widetilde{p}_\varepsilon(x) &\to p(x) \textit{ strongly in } L^2(\Omega). \end{aligned}$$

Remark 47 *Tartar's original work [59] assumes that perforations must be completely separated from each other. See Figure 20. In [3] Allaire's result was extended to include perforations that form a connected lattice.*

4 Reconnecting to the aim and research questions

This chapter contains summaries of the included articles. We state the main results and briefly discuss the methods we have developed.

4.1 Summary of Paper I

In Paper IV [37], we study homogenization of a parabolic linear problem defined on a periodically perforated domain with spatial and temporal oscillations, of the type

$$\begin{aligned} \partial_t u_\varepsilon(x,t) - \nabla \cdot \left(A\left(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon^2}\right) \nabla u_\varepsilon\right) &= f_\varepsilon(x,t) \quad \text{in } \Omega_\varepsilon \times (0,T), \\ u_\varepsilon(x,t) &= 0 \quad \text{on } \partial\Omega \times (0,T), \\ A\left(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon^2}\right) \nabla u_\varepsilon(x,t) \cdot n &= 0 \quad \text{on } (\partial\Omega_\varepsilon - \partial\Omega) \times (0,T), \\ u_\varepsilon(x,0) &= u_\varepsilon^0(x) \quad \text{in } \Omega_\varepsilon, \end{aligned} \tag{16}$$

where

$$\tilde{u}_\varepsilon^0 \rightharpoonup u^0 \text{ in } L^2(\Omega)$$

and

$$\tilde{f}_\varepsilon \rightharpoonup f \text{ in } L^2(\Omega \times (0,T)).$$

Here, $A \in C_\sharp(\mathcal{Y}_{1,1})^{N\times N}$, $f_\varepsilon \in L^2(\Omega_\varepsilon \times (0,T))$ and $u_\varepsilon^0 \in L^2(\Omega_\varepsilon)$. $\tilde{f}_\varepsilon$ and $\tilde{u}_\varepsilon^0$ are extensions with zero to entire Ω. Moreover,

$$A(y,s)\xi \cdot \xi \geq C_0 \left|\xi\right|^2$$

for all $\xi \in \mathbb{R}^N$ and some $C_0 > 0$.

We proved an essential result for the $(2,2)$-scale limit for bounded sequences in $L^2(0,T;H^1(\Omega_\varepsilon))$.

Theorem 48 *(Theorem 20 in Paper I). Assume that $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H^1(\Omega_\varepsilon))$ and for any $v \in D(\Omega)$, $c_1 \in D(0,T)$, $c_2 \in C_\sharp^\infty(0,1)$*

$$\varepsilon^r \int_0^T \int_{\Omega_\varepsilon} u_\varepsilon(x,t) v(x) \partial_t \left(c_1(t) c_2\left(\frac{t}{\varepsilon^r}\right)\right) dxdt \to 0 \tag{17}$$

for some $r > 0$. Then, up to a subsequence,

$$\begin{aligned} &\int_0^T \int_{\Omega_\varepsilon} \nabla u_\varepsilon(x,t) \cdot v(x,t,\frac{x}{\varepsilon},\frac{t}{\varepsilon^r}) dxdt \\ \to\ &\int_0^T \int_\Omega \int_0^1 \int_{Y^*} (\nabla u(x,t) + \nabla_y u_1(x,t,y,s)) \cdot v(x,t,y,s) dyds dxdt \end{aligned}$$

where $u \in L^2(0,T;H^1(\Omega))$, $u_1 \in L^2[\Omega \times (0,T) \times (0,1);\ H^1_\sharp(Y^*)/\mathbb{R})]$, for any $v \in L^2(\Omega \times (0,T); C_\sharp(\mathcal{Y}_{1,1}))$.

In the proof we used the integration by parts to show that u_0 does not depend on y. Then the assumption (17) was used to show that u_0 is independent of s. Further, u_0 was located into the appropriate function space. We integrated by parts and used characterization of the $(2,2)$-scale limit u_0 on one side and the limit w_0 for $\{\nabla u_\varepsilon\}$ on the other. An orthogonality argument was used to find the corrector term $\nabla_y u_1$.

The following is important to handle the time oscillations in the homogenization problem.

Corollary 49 *(Corollary 21 in Paper I) Assume that $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H^1(\Omega_\varepsilon))$ and (17) holds. Then*

$$\int_0^T \int_{\Omega_\varepsilon} \varepsilon^{-1} u_\varepsilon(x,t) v_1(x) v_2\left(\frac{x}{\varepsilon}\right) c_1(t) c_2\left(\frac{t}{\varepsilon^r}\right) dxdt$$
$$\to \int_0^T \int_\Omega \int_0^1 \int_{Y^*} u_1(x,t,y,s) v_1(x) v_2(y) c_1(t) c_2(s) dyds dxdt$$

for $v_1 \in D(\Omega)$, $v_2 \in C^\infty_\sharp(Y^*)/\mathbb{R}$, $c_1 \in D(0,T)$ and $c_2 \in C^\infty_\sharp(0,1)/\mathbb{R}$, where u_1 is the same as in Theorem 48.

In the proof we have expressed v_2 with help of the Poisson equation. We applied the gradient characterization in Theorem 48 for $\{\nabla u_\varepsilon\}$ and created a $(2,2)$-scale limit. We finally used integration by parts, based on the design of v_2.

We carried out a homogenization procedure for (16). The result is found in Theorem 50 below.

Theorem 50 *(Theorem 22 in Paper I) Let $\{u_\varepsilon\}$ be a sequence of solutions to (16). Then it holds that*

$$\widetilde{u}_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} u(x,t)\chi_{Y^*}(y)$$

and

$$\widetilde{\nabla u}_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} (\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y,s))\chi_{Y^*}(y)\,,$$

where (u,u_1) is considered to be the unique solution in $L^2(0,T;H^1_0(\Omega))\times L^2\left[\Omega\times(0,T)\times(0,1);\ H^1_\sharp(Y^*)/\mathbb{R})\right]$ of the following two-scale homogenized system:

$$\begin{aligned}
\mu(Y^*)\partial_t u(x,t) - \nabla\cdot(b\nabla u(x,t)) &= f(x,t) && \text{in } \Omega\times(0,T),\\
u(x,t) &= 0 && \text{on } \partial\Omega\times(0,T),\\
u(x,0) &= (\mu(Y^*))^{-1}u^0(x) && \text{in } \Omega,
\end{aligned}$$

where

$$b\nabla u = \int_0^1 \int_{Y^*} A(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))dyds$$

and u_1 solves our local problem

$$\partial_s u_1(x,t,y,s) - \nabla_y \cdot (A(y,s)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s))) = 0 \text{ in } Y^* \times (0,1)$$

$$(A(y,s)\left[\nabla u(x,t) + \nabla_y u_1(x,t,y,s)\right]) \cdot n = 0 \text{ on } (\partial Y^* - \partial Y) \times (0,1)$$

for a.e. $(x,t) \in \Omega \times (0,T)$.

The proof applies Theorem 48 and Corollary 49 with suitable choices of test functions to obtain the homogenized and the local problems. Moreover, χ_{Y^*} is the characteristic function for Y^*.

4.2 Summary of Paper II

In paper II [36] we study linear parabolic multiscale problems of the type

$$\begin{aligned} \varepsilon\partial_t u_\varepsilon(x,t) - \nabla \cdot \left(a\left(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon^3}\right) \nabla u_\varepsilon(x,t)\right) &= f(x,t) \quad \text{in} \quad \Omega_T, \\ u_\varepsilon(x,0) &= u_0(x) \quad \text{in} \quad \Omega, \\ u_\varepsilon(x,t) &= 0 \qquad \text{on} \quad \partial\Omega \times (0,T), \end{aligned} \tag{18}$$

with a of the same kind as A in Paper I, $u_0 \in L^2(\Omega)$ and $f \in L^2(\Omega_T)$. According to the assumptions on input, the equation (18) possesses a unique solution $u_\varepsilon \in W^{1,2}(0,T; H_0^1(\Omega), L^2(\Omega))$ for a fixed ε but is not uniformly bounded in this space. The main purpose of this paper is to show a convergence of $\{u_\varepsilon\}$ towards a limit and to find an equation, solved by this limit. We proved compactness results for the evolution setting applying the theory of multiscale convergence and established the homogenization result. In the process, we found out that the homogenized problem turns out to be of elliptic type, while the local problem is a parabolic one. The key assumptions in the proof of the necessary compactness results include a certain limit criterion (19). Below we list the main results of the paper.

Theorem 51 *(Theorem 2.7 in Paper II) Assume that $\{u_\varepsilon\}$ is bounded in $L^2(0,T; H_0^1(\Omega))$ and, for any $v_1 \in D(\Omega)$, $c_1 \in D(0,T)$, $c_2 \in C_\sharp^\infty(S)$ and $r > 0$,*

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} u_\varepsilon(x,t) v_1(x) \partial_t \left(\varepsilon^r c_1(t) c_2\left(\frac{t}{\varepsilon^r}\right)\right) dxdt = 0. \tag{19}$$

Then, with $\varepsilon_1 = \varepsilon$ and $\varepsilon_1' = \varepsilon^r$, up to a subsequence,

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \text{ in } L^2(0,T; H_0^1(\Omega)) \tag{20}$$

and

$$\nabla u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} \nabla u(x,t) + \nabla_y u_1(x,t,y,s) \tag{21}$$

where $u \in L^2(0,T; H_0^1(\Omega))$ and $u_1 \in L^2(\Omega_T \times S; H_\sharp^1(Y)/\mathbb{R}))$.

In the theorem above we gave the convergence results for $\{u_\varepsilon\}$ and $\{\nabla u_\varepsilon\}$. To show independence of the local time variable s in u we carried out differentiations in (19). The weak convergence (20) follows from the boundedness of $\{u_\varepsilon\}$ in $L^2(0,T;H_0^1(\Omega))$ and (21) is proved by using the orthogonal reasoning. The following theorem 52 is importaint to handle the rapid time oscillations.

Theorem 52 *(Theorem 2.10 in Paper II) Assume that $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and, for any $v_1 \in D(\Omega)$, $c_1 \in D(0,T)$, $c_2 \in C_\sharp^\infty(S)$ and $r > 0$,*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v_1(x)\partial_t\left(\varepsilon^r c_1(t)c_2\left(\frac{t}{\varepsilon^r}\right)\right)dxdt = 0.$$

Then, with $\varepsilon_1 = \varepsilon$ and $\varepsilon_1' = \varepsilon^r$, up to a subsequence,

$$\varepsilon^{-1}u_\varepsilon(x,t) \overset{2,2}{\underset{vw}{\rightharpoonup}} u_1(x,t,y,s),$$

where $u_1 \in L^2(\Omega_T \times S;\, H_\sharp^1(Y)/\mathbb{R})$ is the same as in (2.3) in Theorem 51.

Note that the asssumption (19) is applied in both theorems to handle the lack of boundedness of $\{\partial_t u_\varepsilon\}$ in $L^2(0,T;H^{-1}(\Omega))$.

The solutions u_ε of (18) are in line with the assertions below and hence Theorems 51 and 52 can be applied to prove the homogenization result in Theorem 53.

(i) $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$, i.e. it satisfies the a priori estimate

$$\|u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \le C,$$

where C is a constant independent of ε.

(ii)

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v_1(x)\partial_t\left(\varepsilon^3 c_1(t)c_2\left(\frac{t}{\varepsilon^3}\right)\right)dxdt = 0,$$

where $v_1 \in D(\Omega)$, $c_1 \in D(0,T)$ and $c_2 \in C_\sharp^\infty(S)$.

The homogenization result which is presented in Theorem 3.2 is shown abbreviated here.

Theorem 53 *Let $\{u_\varepsilon\}$ be a sequence of solutions to (18). Then*

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \text{ in } L^2(0,T;H_0^1(\Omega))$$

and

$$\nabla u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} \nabla u(x,t) + \nabla_y u_1(x,t,y,s),$$

where $u \in L^2(0,T;H_0^1(\Omega))$ is the unique solution to

$$\begin{aligned} -\nabla\cdot(b\nabla u(x,t)) &= f(x,t) \text{ in } \Omega_T, \\ u(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,T), \end{aligned}$$

with

$$b\nabla u(x,t) = \int_{\mathcal{Y}_{1,1}} a(y,s)\left(\nabla u(x,t) + \nabla_y u_1(x,t,y,s)\right) dyds.$$

Here, u_1 is the unique solution to the local problem

$$\partial_s u_1(x,t,y,s) - \nabla_y \cdot \left(a(y,s)\left(\nabla u(x,t) + \nabla_y u_1(x,t,y,s)\right)\right) = 0.$$

Here, $u_1 \in L^2(\Omega_T, \mathcal{W})$. The existence of weak solutions to the homogenized and local problems is displayed via the homogenization procedure.

4.3 Summary of Paper III

In Paper III a study of homogenization of the heat conduction equation is performed for a homogeneous Dirichlet boundary condition with oscillating thermal conductivity and vanishing volumetric heat capacity. We investigate different matchings between the coefficient ε^q, the spatial period ε and the period ε^r for the time oscillations.

The focus of this paper is on the solution of the equation below

$$\begin{aligned} \varepsilon^q \partial_t u_\varepsilon(x,t) - \nabla\cdot\left(a\left(\tfrac{x}{\varepsilon}, \tfrac{t}{\varepsilon^r}\right)\nabla u_\varepsilon(x,t)\right) &= f(x,t) \quad \text{in} \quad \Omega\times(0,T), \\ u_\varepsilon(x,0) &= u_0(x) \quad \text{in} \quad \Omega, \\ u_\varepsilon(x,t) &= 0 \quad \text{on} \quad \partial\Omega\times(0,T), \end{aligned} \tag{22}$$

when $0 < q < r$ and the assumptions on a, f and u_0 are the same as in Paper II. As $\varepsilon \to 0$, we found how the relation between the volumetric heat capacity and the microscopic structure governed by q and r affects the limit problem and its related local problem. Here, u is the unique solution to the homogenized problem

$$\begin{aligned} -\nabla\cdot(b\nabla u(x,t)) &= f(x,t) \quad \text{in} \quad \Omega_T, \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega\times(0,T). \end{aligned}$$

We state here Theorem 2 from Paper III.

Theorem 54 *(Theorem 2 in Paper III) Assume that $\{u_\varepsilon\}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and, for any $v_1 \in C_0^\infty(\Omega)$, $c_1 \in C_0^\infty(0,T)$, $c_2 \in C_\sharp^\infty(S)$ and $r > 0$*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)v_1(x)\partial_t\left(\varepsilon^r c_1(t)c_2\left(\frac{t}{\varepsilon^r}\right)\right)dxdt = 0.$$

Then, for $n = m = 1$ with $\varepsilon_1 = \varepsilon$ and $\varepsilon_1' = \varepsilon^r$, up to a subsequence,

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \text{ in } L^2(0,T;H_0^1(\Omega)),$$

$$\nabla u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} \nabla u(x,t) + \nabla_y u_1(x,t,y,s)$$

and

$$\varepsilon^{-1}u_\varepsilon(x,t) \overset{2,2}{\underset{vw}{\rightharpoonup}} u_1(x,t,y,s),$$

where $u \in L^2(0,T;H_0^1(\Omega))$ and $u_1 \in L^2(\Omega_T \times S; H_\sharp^1(Y)/\mathbb{R})$.

Proof. See Theorem 2.7 and Theorem 2.10 in [36]. ■

This theorem represents a gradient characterization and a compactness result for very weak evolution multiscale convergence. In the homogenization result of Theorem 3 in Paper III, we're engaged in finding local problems for the three cases applying Theorem 54. We list here all three of them in the following theorem.

Theorem 55 *(Theorem 3 in Paper III) Let $\{u_\varepsilon\}$ be a sequence of solutions to (22) in $W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$. Then it holds that*

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \text{ in } L^2(0,T;H_0^1(\Omega))$$

and

$$\nabla u_\varepsilon(x,t) \overset{2,2}{\rightharpoonup} \nabla u(x,t) + \nabla_y u_1(x,t,y,s),$$

where $u \in L^2(0,T;H_0^1(\Omega))$ and $u_1 \in L^2(\Omega_T \times S; H_\sharp^1(Y)/\mathbb{R})$. Here, u is the unique solution to the homogenized problem

$$\begin{aligned} -\nabla \cdot (b\nabla u(x,t)) &= f(x,t) \quad \text{in} \quad \Omega_T, \\ u(x,t) &= 0 \quad \text{on} \quad \partial\Omega \times (0,T), \end{aligned}$$

with, for $q < r < q+2$,

$$b\nabla u(x,t) = \int_{\mathcal{Y}_{1,1}} a(y,s)\left(\nabla u(x,t) + \nabla_y u_1(x,t,y,s)\right) dyds \tag{23}$$

where $u_1 \in L^2(\Omega_T \times S; H_\sharp^1(Y)/\mathbb{R})$ is determined by the elliptic local problem

$$\nabla_y \cdot \left(a(y,s)\left(\nabla u(x,t) + \nabla_y u_1(x,t,y,s)\right)\right) = 0,$$

for $r = q+2$, $b\nabla u(x,t)$ is given by (23), where $u_1 \in L^2(\Omega_T; \mathcal{W})$ is determined by the parabolic local problem

$$\partial_s u_1(x,t,y,s) - \nabla_y \cdot \left(a(y,s)\left(\nabla u(x,t) + \nabla_y u_1(x,t,y,s)\right)\right) = 0.$$

and, for $r > q+2$,

$$b\nabla u(x,t) = \int_Y \left(\int_S a(y,s)ds\right)\left(\nabla u(x,t) + \nabla_y u_1(x,t,y)\right) dy$$

where $u_1 \in L^2(\Omega_T; H_\sharp^1(Y)/\mathbb{R})$ is given by the elliptic local problem

$$-\nabla_y \cdot \left(\left(\int_S a(y,s)ds\right)\left(\nabla u(x,t) + \nabla_y u_1(x,t,y)\right)\right) = 0.$$

The outcome of the homogenization procedure is two special features. For all $0 < q < r$ we got the elliptic homogenized problem and, for different matching between the microscopic scales, a parabolic or an elliptic local problem. For a more detailed treatment, see [21].

4.4 Summary of Paper IV

This paper is a continuation of the work carried out in Papers II and III. The main purpose of this work is focused on establishing relevant compactness results and homogenization results for the equation below. The references for the initial ideas were Lobkova, [37] and Johnsen and Lobkova [36].

We consider the homogenization of the linear parabolic equation

$$\begin{aligned} \varepsilon^p \partial_t u_\varepsilon(x,t) - \nabla\cdot\big(a\left(\tfrac{x}{\varepsilon}, \tfrac{x}{\varepsilon^2}, \tfrac{t}{\varepsilon^q}, \tfrac{t}{\varepsilon^r}\right)\nabla u_\varepsilon(x,t)\big) &= f(x,t) \quad \text{in } \Omega_T, \\ u_\varepsilon(x,0) &= u_0(x) \quad \text{in } \Omega, \\ u_\varepsilon(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,T). \end{aligned} \tag{24}$$

We deal with $0 < p < q < r$, $f \in L^2(\Omega_T)$ and $u_0 \in L^2(\Omega)$. The coefficient $a \in C_\sharp(\mathcal{Y}_{2,2})^{N\times N}$ is assumed to be periodic with coercivity condition

$$a(y^2, s^2)\xi\cdot\xi \geq C_0 \left|\xi\right|^2$$

for all $(y^2, s^2) \in \mathbb{R}^{2N}\times\mathbb{R}^2$, all $\xi \in \mathbb{R}^N$ and $C_0 > 0$.

We introduce assumptions corresponding to (19) adapted to (24) in addition to $\{u_\varepsilon\}$ being bounded in $L^2(0,T;H_0^1(\Omega))$. However, we generalize the result from [36] to the $(2,3)$-scale and $(3,3)$-scale convergence types. We state and prove Theorems 56 and 57, where we find characterizations of the $(2,3)$-scale and $(3,3)$-scale limits for $\{\nabla u_\varepsilon\}$, under certain assumptions and consider very weak $(2,3)$-scale and $(3,3)$-scale convergence for $\left\{\varepsilon^{-1}u_\varepsilon\right\}$ and $\left\{\varepsilon^{-2}u_\varepsilon\right\}$, respectively.

Theorem 56 *(Theorem 6 in Paper IV) Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(0,T;H_0^1(\Omega))$ and, for any $v \in D(\Omega)$, $c_1 \in D(0,T)$, $c_2 \in C_\sharp^\infty(S_1)$, $c_3 \in C_\sharp^\infty(S_2)$ and $r > q > 0$,*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)\, v(x)\, \partial_t\left(\varepsilon^r c_1(t)\, c_2\left(\frac{t}{\varepsilon^q}\right) c_3\left(\frac{t}{\varepsilon^r}\right)\right) dxdt = 0$$

and

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)\, v(x)\, \partial_t\left(\varepsilon^q c_1(t)\, c_2\left(\frac{t}{\varepsilon^q}\right)\right) dxdt = 0.$$

Then, with $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$, $\varepsilon_1' = \varepsilon^q$ and $\varepsilon_2' = \varepsilon^r$, up to a subsequence,

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \ \text{ in } L^2(0,T;H_0^1(\Omega)),$$

$$u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} u(x,t),$$

$$\nabla u_\varepsilon(x,t) \overset{2,3}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1\left(x,t,y_1,s^2\right) \tag{25}$$

and

$$\nabla u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1\left(x,t,y_1,s^2\right) + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right), \tag{26}$$

where $u \in L^2(0,T;H_0^1(\Omega))$, $u_1 \in L^2(\Omega_T\times S^2; H_\sharp^1(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T\times\mathcal{Y}_{1,2}; H_\sharp^1(Y_2)/\mathbb{R})$.

As a consequence, the continuation of the theorem is a proof of compactness results for very weak evolution multiscale convergence for $\{u_\varepsilon\}$ without requiring boundedness of the time derivative in $L^2(0,T;H^{-1}(\Omega))$.

Theorem 57 *(Theorem 9 in Paper IV) Let $\{u_\varepsilon\}$ be a bounded sequence in $L^2(0,T;H_0^1(\Omega))$ and, for any $v \in D(\Omega)$, $c_1 \in D(0,T)$, $c_2 \in C_\sharp^\infty(S_1)$, $c_3 \in C_\sharp^\infty(S_2)$ and $r > q > 0$,*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)\, v(x)\, \partial_t\left(\varepsilon^r c_1(t)\, c_2\left(\frac{t}{\varepsilon^q}\right) c_3\left(\frac{t}{\varepsilon^r}\right)\right) dxdt = 0$$

and

$$\lim_{\varepsilon\to 0}\int_{\Omega_T} u_\varepsilon(x,t)\, v(x)\, \partial_t\left(\varepsilon^q c_1(t)\, c_2\left(\frac{t}{\varepsilon^q}\right)\right) dxdt = 0.$$

Then, with $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon^2$, $\varepsilon_1' = \varepsilon^q$ and $\varepsilon_2' = \varepsilon^r$, up to a subsequence

$$\varepsilon^{-1} u_\varepsilon(x,t) \overset{2,3}{\underset{vw}{\rightharpoonup}} u_1\left(x,t,y_1,s^2\right)$$

and

$$\varepsilon^{-2} u_\varepsilon(x,t) \overset{3,3}{\underset{vw}{\rightharpoonup}} u_2\left(x,t,y^2,s^2\right),$$

where $u_1 \in L^2(\Omega_T \times S^2; H_\sharp^1(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H_\sharp^1(Y_2)/\mathbb{R})$ are the same as in (25) and (26) in Theorem 56.

We deduce the following theorem using information on the choices of p, q and r in (24), so we get 13 possible outcomes.

Theorem 58 *(Theorem 10 in Paper IV) Let $\{u_\varepsilon\}$ be a sequence of solutions to (24) in $W^{1,2}(0,T;H_0^1(\Omega),L^2(\Omega))$. Then it holds that*

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \ \text{ in } L^2(0,T;H_0^1(\Omega))$$

$$u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} u(x,t)$$

and

$$\nabla u_\varepsilon(x,t) \overset{3,3}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1\left(x,t,y_1,s^2\right) + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right),$$

where $u \in L^2(0,T;H_0^1(\Omega))$ is the unique solution to the homogenized problem

$$\begin{aligned} -\nabla\cdot\left(b\nabla u(x,t)\right) &= f(x,t) \ \text{ in } \Omega_T, \\ u(x,t) &= 0 \ \text{ on } \partial\Omega\times(0,T), \end{aligned}$$

where the coefficient b is characterized by the formulas below. For all 13 cases we assume that $0 < p < q < r$.

1. *Letting $r < 2 + p$, the homogenized coefficient is given by*

$$b\nabla u\left(x,t\right) = \int_{\mathcal{Y}_{2,2}} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s^2\right)\right. \\ \left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy^2 ds^2, \tag{27}$$

and $u_1 \in L^2(\Omega_T \times S^2; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H^1_\sharp(Y_2)/\mathbb{R})$ are given by the local problems

$$-\nabla_{y_2} \cdot \left(a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s^2\right)\right.\right. \\ \left.\left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right)\right) = 0$$

and

$$-\nabla_{y_1} \cdot \int_{Y_2} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s^2\right)\right. \\ \left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy_2 = 0.$$

2. *Choosing $r = 2 + p$, the coefficient b is determined by (27) while $u_1 \in L^2(\Omega_T \times S_1; \mathcal{W}_{1,2})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H^1_\sharp(Y_2)/\mathbb{R})$ are the solutions to the local problems*

$$-\nabla_{y_2} \cdot \left(a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s^2\right)\right.\right. \\ \left.\left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right)\right) = 0$$

and

$$\partial_{s_2} u_1\left(x,t,y_1,s^2\right) - \nabla_{y_1} \cdot \int_{Y_2} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s^2\right)\right. \\ \left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy_2 = 0.$$

3. *If $2 + p < r < 4 + p$ while $q < 2 + p$, we have*

$$b\nabla u\left(x,t\right) = \int_{\mathcal{Y}_{2,2}} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right)\right. \\ \left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy^2 ds^2 \tag{28}$$

where $u_1 \in L^2(\Omega_T \times S_1; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H^1_\sharp(Y_2)/\mathbb{R})$ are given by the system

$$-\nabla_{y_2} \cdot \left(a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right)\right.\right. \\ \left.\left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right)\right) = 0$$

and

$$\nabla_{y_1} \cdot \int_{Y_2 \times S_2} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right)\right. \\ \left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy_2 ds_2 = 0.$$

4. *Taking $r < 4+p$ and $q = 2+p$, the homogenized coefficient is given by (28) and $u_1 \in L^2(\Omega_T; \mathcal{W}_{1,1})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H^1_\sharp(Y_2)/\mathbb{R})$ are determined by*

$$-\nabla_{y_2} \cdot \left(a\left(y^2, s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right)\right.\right.$$
$$\left.\left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right)\right) = 0$$

and

$$\partial_{s_1} u_1\left(x,t,y_1,s_1\right) - \nabla_{y_1} \cdot \int_{Y_2 \times S_2} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right)\right.$$
$$\left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy_2 ds_2 = 0.$$

5. *When $r < 4+p$ and $q > 2+p$ the coefficient b is determined by*

$$b\nabla u\left(x,t\right) = \int_{\mathcal{Y}_{2,2}} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1\right)\right. \tag{29}$$
$$\left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy^2 ds^2$$

and the local problems are

$$-\nabla_{y_2} \cdot \left(a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1\right)\right.\right.$$
$$\left.\left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right)\right) = 0$$

and

$$-\nabla_{y_1} \cdot \int_{Y_2 \times S^2} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1\right)\right.$$
$$\left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy_2 ds^2 = 0,$$

where $u_1 \in L^2(\Omega_T; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}; H^1_\sharp(Y_2)/\mathbb{R})$.

6. *In the case when $r = 4+p$ while $q < 2+p$, the homogenized coefficient is characterized by (28) while $u_1 \in L^2(\Omega_T \times S_1; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; \mathcal{W}_{2,2})$ are given by the system of local problems*

$$\partial_{s_2} u_2\left(x,t,y^2,s^2\right) - \nabla_{y_2} \cdot \left(a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right)\right.\right.$$
$$\left.\left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right)\right) = 0$$

and

$$-\nabla_{y_1} \cdot \int_{Y_2 \times S_2} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right)\right.$$
$$\left. + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy_2 ds_2 = 0.$$

7. *When $r = 4 + p$ and $q = 2 + p$, the coefficient b is given by (28) where $u_1 \in L^2(\Omega_T; \mathcal{W}_{1,1})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; \mathcal{W}_{2,2})$ are the solutions to*

$$\partial_{s_2} u_2\left(x,t,y^2,s^2\right) - \nabla_{y_2} \cdot \left(a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right) + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right)\right) = 0$$

and

$$\partial_{s_1} u_1\left(x,t,y_1,s_1\right) - \nabla_{y_1} \cdot \int_{Y_2 \times S_2} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right) + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy_2 ds_2 = 0.$$

8. *Letting $r = 4 + p$ while $q > 2 + p$ gives us the homogenized coefficient (29) defined by the system of local problems*

$$\partial_{s_2} u_2\left(x,t,y^2,s^2\right) - \nabla_{y_2} \cdot \left(a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1\right) + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right)\right) = 0$$

and

$$-\nabla_{y_1} \cdot \int_{Y_2 \times S^2} a\left(y^2,s^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1\right) + \nabla_{y_2} u_2\left(x,t,y^2,s^2\right)\right) dy_2 ds^2 = 0,$$

where $u_1 \in L^2(\Omega_T; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; \mathcal{W}_{2,2})$.

9. *Choosing $r > 4 + p$ and $q < 2 + p$, we have the homogenized coefficient*

$$b\nabla u\left(x,t\right) = \int_{\mathcal{Y}_{2,1}} \left(\int_{S_2} a\left(y^2,s^2\right) ds_2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right) + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right)\right) dy^2 ds_1 \tag{30}$$

where $u_1 \in L^2(\Omega_T \times S_1; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; H^1_\sharp(Y_2)/\mathbb{R})$ are the solutions to the local problems

$$-\nabla_{y_2} \cdot \left(\int_{S_2} a\left(y^2,s^2\right) ds_2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right) + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right)\right) = 0$$

and

$$-\nabla_{y_1} \cdot \int_{Y_2} \left(\int_{S_2} a\left(y^2,s^2\right) ds_2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right) + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right)\right) dy_2 = 0.$$

10. *When $r > 4+p$ while $q = 2+p$, the homogenized coefficient is given by (30) and the local problems are*

$$-\nabla_{y_2} \cdot \left(\int_{S_2} a\left(y^2, s^2\right) ds_2 \right) \left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right) \right.$$
$$\left. + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right) \right) = 0$$

and

$$\partial_{s_1} u_1\left(x,t,y_1,s_1\right) - \nabla_{y_1} \cdot \int_{Y_2} \left(\int_{S_2} a\left(y^2, s^2\right) ds_2 \right) \left(\nabla u\left(x,t\right) \right.$$
$$\left. + \nabla_{y_1} u_1\left(x,t,y_1,s_1\right) + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right) \right) dy_2 = 0,$$

with $u_1 \in L^2(\Omega_T; \mathcal{W}_{1,1})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; H^1_\sharp(Y_2)/\mathbb{R})$.

11. *When $r > 4+p$ and $2+p < q < 4+p$, we have*

$$b\nabla u\left(x,t\right) = \int_{\mathcal{Y}_{2,1}} \left(\int_{S_2} a\left(y^2, s^2\right) ds_2 \right) \left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1\right) \right. \quad (31)$$
$$\left. + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right) \right) dy^2 ds_1$$

together with the local problems

$$-\nabla_{y_2} \cdot \left(\int_{S_2} a\left(y^2, s^2\right) ds_2 \right) \left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1\right) \right.$$
$$\left. + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right) \right) = 0$$

and

$$-\nabla_{y_1} \cdot \int_{Y_2 \times S_1} \left(\int_{S_2} a\left(y^2, s^2\right) ds_2 \right) \left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1\right) \right.$$
$$\left. + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right) \right) dy_2 ds_1 = 0,$$

where $u_1 \in L^2(\Omega_T; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; H^1_\sharp(Y_2)/\mathbb{R})$.

12. *Taking $q = 4+p$, the coefficient in the homogenized problem is given by (31) and $u_1 \in L^2(\Omega_T; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times Y_1; \mathcal{W}_{2,1})$ are determined by*

$$\partial_{s_1} u_2\left(x,t,y^2,s_1\right) - \nabla_{y_2} \cdot \left(\int_{S_2} a\left(y^2, s^2\right) ds_2 \right) \left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1\right) \right.$$
$$\left. + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right) \right) = 0$$

and

$$-\nabla_{y_1} \cdot \int_{Y_2 \times S_1} \left(\int_{S_2} a\left(y^2, s^2\right) ds_2 \right) \left(\nabla u\left(x,t\right) + \nabla_{y_1} u_1\left(x,t,y_1\right) \right.$$
$$\left. + \nabla_{y_2} u_2\left(x,t,y^2,s_1\right) \right) dy_2 ds_1 = 0.$$

13. In the case when $q > 4 + p$, the coefficient is characterized by

$$b\nabla u\left(x,t\right) = \int_{Y^2}\left(\int_{S^2} a\left(y^2,s^2\right)ds^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1}u_1\left(x,t,y_1\right) + \nabla_{y_2}u_2\left(x,t,y^2\right)\right)dy^2$$

and the local problems are given by

$$-\nabla_{y_2}\cdot\left(\int_{S^2} a\left(y^2,s^2\right)ds^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1}u_1\left(x,t,y_1\right) + \nabla_{y_2}u_2\left(x,t,y^2\right)\right) = 0$$

and

$$-\nabla_{y_1}\cdot\int_{Y_2}\left(\int_{S^2} a\left(y^2,s^2\right)ds^2\right)\left(\nabla u\left(x,t\right) + \nabla_{y_1}u_1\left(x,t,y_1\right) + \nabla_{y_2}u_2\left(x,t,y^2\right)\right)dy_2 = 0,$$

where $u_1 \in L^2(\Omega_T; H^1_\sharp(Y_1)/\mathbb{R})$ and $u_2 \in L^2(\Omega_T \times Y_1; H^1_\sharp(Y_2)/\mathbb{R})$.

4.5 Summary of Paper V

In paper V we consider the homogenization of a monotone parabolic problem with an arbitrary number of microscopic scales in space and time. This means that we do not require the problem to be linear. Moreover, a particular point is that the scale functions do not necessarily need to be powers of ε. We studied the homogenization of the problem

$$\begin{aligned} \partial_t u^\varepsilon(x,t) - \nabla\cdot a\left(\tfrac{x}{\varepsilon_1},\cdots,\tfrac{x}{\varepsilon_n},\tfrac{t}{\varepsilon'_1},\cdots,\tfrac{t}{\varepsilon'_m},\nabla u^\varepsilon\right) &= f(x,t) \text{ in } \Omega_T,\\ u^\varepsilon(x,0) &= u^0(x) \text{ in } \Omega,\\ u^\varepsilon(x,t) &= 0 \text{ on } \partial\Omega\times(0,T), \end{aligned} \tag{32}$$

where $f \in L^2(\Omega_T)$ and $u^0 \in L^2(\Omega)$. We assumed that the lists $\{\varepsilon_1,\ldots,\varepsilon_n\}$ and $\{\varepsilon'_1,\ldots,\varepsilon'_m\}$ in (32) are jointly well-separated and imposed suitable conditions of monotonicity and periodicity on the function a.

We proved the following theorem.

Theorem 59 *(Theorem 9 in Paper V). Let $\{u^\varepsilon\}$ be a sequence of solutions in $W^1_2(0,T;H^1_0(\Omega),L^2(\Omega))$ to (32). Then it holds that*

$$u^\varepsilon(x,t) \to u(x,t) \text{ in } L^2(\Omega_T),$$

$$u^\varepsilon(x,t) \rightharpoonup u(x,t) \text{ in } L^2(0,T;H^1_0(\Omega))$$

and

$$\nabla u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^{n}\nabla_{y_j}u_j\left(x,t,y^j,s^{m-d_j}\right),$$

where $u \in W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ *is the unique solution to*

$$\begin{aligned} \partial_t u(x,t) - \nabla \cdot b(x,t,\nabla u) &= f(x,t) \quad in \quad \Omega_T, \\ u(x,0) &= u^0(x) \quad in \quad \Omega, \\ u(x,t) &= 0 \quad on \quad \partial\Omega \times (0,T), \end{aligned}$$

with

$$\begin{aligned} &b(x,t,\nabla u(x,t)) \\ &= \int_{\mathcal{Y}_{n,m}} a\left(y^n, s^m, \nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j(x,t,y^j,s^{m-d_j})\right) dy^n ds^m, \end{aligned}$$

where $u_i \in L^2(\Omega_T \times \mathcal{Y}_{i-1,m-d_i}; H^1_\sharp(Y_i)/\mathbb{R})$ *for* $i = 1,\dots,n$. *Here* u_i, *for* $i = 1,\dots,n$, *are the unique solutions to the system of local problems*

$$\begin{aligned} &\rho_i \partial_{s_{m-d_i}} u_i(x,t,y^i,s^{m-d_i}) \\ -\nabla_{y_i} \cdot \int_{S_{m-d_i+1}} \cdots \int_{S_m} \int_{Y_{i+1}} \cdots \int_{Y_n} &a\left(y^n, s^m, \nabla u(x,t) + \sum_{j=1}^{n} \nabla_{y_j} u_j\big(x,t,y^j,s^{m-d_j}\big)\right) \\ &\times dy_n \cdots dy_{i+1} ds_m \cdots ds_{m-d_i+1} = 0 \end{aligned}$$

if we assume that $u_i \in L^2(\Omega_T \times \mathcal{Y}_{i-1,m-d_i-1}; W^1_{2_\sharp}(S_{m-d_i}; H^1_\sharp(Y_i)/\mathbb{R}, L^2_\sharp(Y_i)/\mathbb{R}))$ *when* $\rho_i \neq 0$. *Here* d_i *is the number of temporal scales faster than the square of the spatial scale in question and* ρ_i *indicates whether there is resonance (which occurs when the square of the i-th spatial scale coincides with one of the temporal scales [30]) or not.*

The proof was performed by using techniques such as multiscale convergence and very-weak multiscale convergence adapted to evolution problems. See e.g. [30] and [48]. For the characterization of b we used perturbed test functions (see [24] and [25]). In the terminology of paper IV and the list of notations $W^1_{2_\sharp}(S_{m-d_i}; H^1_\sharp(Y_i)/\mathbb{R}, L^2_\sharp(Y_i)/\mathbb{R})) = \mathcal{W}_{i,m-d_i}$.

4.6 Summary of Paper VI

In Paper VI, which is currently at manuscript stage, we study and compare essential properties of and the relation between two-scale convergence and unfolding methods. The author's main contribution in this paper is the proof of Proposition 9 in Section 6.1. The purpose was to show the link between a generalization of two-scale convergence and unfolding. Two-scale convergence can be seen as a special case of a limit process governed by two-scale operators, see e.g. [32], [60], [34], [35], [51]. This concept was named by operator-governed two-scale convergence in [47]. We state the definition.

Definition 60 *Let $\{\gamma_h\}$ be a sequence of linear operators*

$$\gamma_h : X \to L^2(\Omega),$$

where $X \subset L^2(\Omega \times Y)$. A sequence $\{u_h\}$ in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0 \in L^2(\Omega \times Y)$ with respect to $\{\gamma_h\}$ if

$$\lim_{h\to\infty} \int_\Omega u_h(x)\, \gamma_h(v)(x)\, dx = \int_\Omega \int_Y u_0(x,y)\, v(x,y)\, dydx$$

for all $v \in X$.

Here, $\{\gamma_h\}$ are two-scale operators and X is the space of admissible test functions. To specify properties of two-scale operators $\{\gamma_h\}$ we needed the following two definitions.

Definition 61 *A sequence of operators $\{\gamma_h\}$ is said to be two-scale compatible with respect to $X \subset L^2(\Omega \times Y)$, where X is a normed space, if*

$$\gamma_h : X \to L^2(\Omega)$$

are linear,

$$\|\gamma_h(v)\|_{L^2(\Omega)} \leq C \|v\|_X \tag{33}$$

and

$$\lim_{h\to\infty} \|\gamma_h(v)\|_{L^2(\Omega)} \leq D \|v\|_{L^2(\Omega\times Y)} \tag{34}$$

for all $v \in X$.

Definition 62 *Let $\{u_h\}$ be a bounded sequence in $L^2(\Omega)$ two-scale converging to u_0 with respect to $\{\gamma_h\}$, that is two-scale compatible with respect to X. We say that $\{\gamma_h\}$ is strongly two-scale compatible with respect to X if the conditions (33) and (34) together with the two requirements below are fulfilled:*

$$u_h(x) \rightharpoonup \int_Y u_0(x,y)\, dy \text{ in } L^2(\Omega), \tag{35}$$

and

$$\gamma_h(v) \rightharpoonup \int_Y v(x,y)\, dy \text{ in } L^2(\Omega), \tag{36}$$

for all v in X.

The result in Theorem 63 below holds also for other choices of bounded open sets Y than the unit cube. See [51] or [34]. We let, in general, Y be the unit cube in our treatment for the sake of simplicity.

Theorem 63 *Let $\{u_h\}$ be a bounded sequence in $L^2(\Omega)$ and $\{\gamma_h\}$ be strongly two-scale compatible with respect to X, where $X \subset L^2(\Omega \times Y)$ is a separable Banach space. Then there exists a subsequence to $\{u_h\}$ which two-scale converges with respect to $\{\gamma_h\}$ to some $u_0 \in L^2(\Omega \times Y)$ and (35) and (36) are fulfilled.*

An adjoint operator $\mathcal{U}_\varepsilon$ to the unfolding operator τ_ε such that

$$\int_\Omega \int_Y \tau_{\varepsilon_h}(u)(x,y)v(x,y)dydx = \int_\Omega u(x)\mathcal{U}_{\varepsilon_h}(v)(x)dx$$

for any $u \in L^2(\Omega)$, $v \in L^2(\Omega \times Y)$, when $|Y| = 1$ is defined in Definition 2.16 in [13].

We proved the following proposition.

Proposition 64 *Let $\{u_{\varepsilon_h}\}$ be a bounded sequence in $L^2(\Omega)$. Then there exists a subsequence that two-scale converges with respect to $\{\mathcal{U}_{\varepsilon_h}\}$ for $X = L^2(\Omega \times Y)$ and (35) and (36) are fulfilled for $\{u_{\varepsilon_h}\}$ and $\{\mathcal{U}_{\varepsilon_h}\}$, respectively, and the limit is the same as for unfolding and periodic two-scale convergence.*

The proof was performed in a couple of steps. The first step is to show the existence of a two-scale limit with respect to $\mathcal{U}_\varepsilon$ for $X = L^2(\Omega \times Y)$ using the boundedness of $\mathcal{U}_\varepsilon$ as an operator

$$\mathcal{U}_\varepsilon : L^2(\Omega \times Y) \to L^2(\Omega).$$

Then we used basic properties of τ_ε and $\mathcal{U}_\varepsilon$ to show (35) and (36) for $\gamma_h = \mathcal{U}_{\varepsilon_h}$.

Remark 65 *Note also that classical two-scale convergence defined by*

$$\gamma_h(v)(x) = v\left(x, \frac{x}{\varepsilon_h}\right),$$

where $\varepsilon_h \to 0$ for $h \to \infty$, is strongly two-scale compatible with respect to $X = L^2(\Omega; C_\sharp(Y))$.

5 Conclusions

The aim of this dissertation has been to further develop convergence results for homogenization problems with various combinations of scales. It includes a summary of six papers, which was divided into three different groups based on the stated sub-aims. We summarize the main findings from the set of papers:

Paper I answers aim 1 and concludes the following:

- The method of two-scale convergence type introduced in [2] for periodically perforated media can be adapted to time-dependent problems, which means that we develop a method without nontrivial extension techniques.

Papers II, III, IV and V answer aims 2 and 3 and conclude the following:

- The appearance of the small parameter ε in front of the time derivative led to the fact that the homogenized equation appeared to be an elliptic one. The local problem becomes parabolic in Paper II, but in Paper III and IV, the nature of the local problem depends on the matching between the scales.

  This means that the relationship between the coefficient passing to zero that is multiplied with the time derivative term and the spatial and temporal microoscillations in the parabolic equation strongly affect the homogenized and local problem.

- We examined possible extensions of achieved results in [28] and [30] and designed a homogenization procedure for nonlinear parabolic problems with multiple spatial and temporal scales, where the oscillating operator is monotone and not necessarily linear.

Papers VI answer aim 4 and concludes the following:

- The comparison between the essential properties of two-scale convergence and unfolding leads to a new way to understand the relation between unfolding and periodic two-scale convergence.

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