Two-dimensional slices of nonpseudoconvex domains with rough boundary

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Abstract For a domain $D \subset \mathbb{C}^n$, $n \geq 3$, the set E is defined as the set of all points $z \in \mathbb{C}^n$ for which the intersection of D with every complex 2-plane through z is pseudoconvex. For D nonpseudoconvex, it is shown that E is contained in an affine subspace of codimension 2. This results solves a problem raised by Nikolov and Pflug.

Keywords Pseudoconvexity · Sections of nonpseudoconvex domains · Continuity principle

Mathematics Subject Classification 32T05 · 32D20 · 32D26 · 32A40

1 Introduction

It was observed by Hitotumatu [2], see also [5], that a domain $D \subset \mathbb{C}^n$, $n \geq 3$, is pseudoconvex iff its intersections with all 2-dimensional affine complex planes P are pseudoconvex open sets. On the other hand, the family of planes passing through a single point does not suffice to detect nonpseudoconvexity. For a nonpseudoconvex domain $D \subset \mathbb{C}^n$, we call $z \in \mathbb{C}^n$ *exceptional* if the intersection $P \cap D$ is pseudoconvex for every 2-plane P passing through z (the empty set is considered as pseudoconvex). We denote the set of exceptional points by E . As an example for a large E , we may consider a pseudoconvex $G \subset \mathbb{C}^3$ and a line $\ell \subset \mathbb{C}^3$ with $\ell \cap G \neq \emptyset$, in which case the points in $\ell \setminus G$ are exceptional for $D = G \setminus \ell$, see [7]. In general the set E cannot be too large: In [7] it was proved that the set E of exceptional points is always contained in an affine complex hyperplane of $\mathbb{C}^n$.

For domains with higher boundary regularity, a finer study of E is given in [6]. There it is shown for certain domains in $\mathbb{C}^3$ that E is contained in a line. Moreover the question is raised whether this holds for every domain in $\mathbb{C}^3$ [6, Remark 3 (1)]. Our main result answers the question in the affirmative.

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Theorem 1.1 *Let $D \subset \mathbb{C}^n$, $n \geq 3$, be a nonpseudoconvex domain. Then the set E of exceptional points is contained in an affine complex subspace of codimension two.*

The above-mentioned example shows that the theorem is sharp. If one assumes that ∂D is more regular, stronger conclusions are possible [6]: For example, E is empty for a nonpseudoconvex D with C^2 -boundary.

2 Proof of the main result

Assume that E contains n points $p_0, p_1, \dots, p_{n-1}$ which do not lie in a common affine subspace of dimension $n - 2$. The theorem will be proved as soon as we have deduced that D is pseudoconvex. After an affine coordinate change, we can assume that $p_0, p_1, \dots, p_{n-1}$ are contained in $H = \{z_n = 0\}$. We start by

Lemma 2.1 *The complement $D_1 = D \setminus H$ is a pseudoconvex domain.*

Proof of Lemma 2.1 Observe that $p_0, p_1, \dots, p_{n-1}$ are also exceptional points of D_1 . Indeed, for a plane P through p_j which is not contained in H the intersection $P \cap D_1$ equals $P \cap D$ minus a complex line, which is pseudoconvex or empty.

Assume that D_1 is not pseudoconvex. As pointed out in [7, Proposition 3], there is a point $q \in \partial D_1$, affine coordinates $w_1 = u_1 + i v_1, \dots, w_n = u_n + i v_n$ centered at q and constants $\eta > 0$, $c < 1$, such that

$$G = \{w \in \mathbb{C}^n : u_n < u_1^2 - c v_1^2 - |w_2|^2 - \dots - |w_{n-1}|^2 - v_n^2, |w| < \eta\}$$

is contained in D_1 . Now $c < 1$ means that $\partial G \cap \{w_2 = \dots = w_{n-1} = 0\}$ is strictly pseudoconcave at q in the (w_1, w_n) -plane. We claim that $q \notin H$. Otherwise the hyperplane H , which lies in the complement of $G \subset D_1$, would coincide with the complex tangent space $T_q = T_q^c \partial G$ of ∂G at q (viewed as affine hyperplane of $\mathbb{C}^n$). Thus $T_q \cap \{w_2 = \dots = w_{n-1} = 0\}$ would touch $G \cap \{w_2 = \dots = w_{n-1} = 0\}$ from outside, in contradiction to strict pseudoconcavity.

Next we claim that the exceptional points of D_1 are contained in T_q . The reason is essentially contained in [7, Proposition 13] and its proof: For $q' \notin T_q$, we consider the plane $P = q + \text{span}_{\mathbb{C}}\{q - q', e_1\}$, where $e_1 = (1, 0, \dots, 0)$ is taken in the coordinates $w_1, \dots, w_n$. Then q is a strictly pseudoconcave point of $G \cap P$, meaning that $D_1 \cap P$ cannot be pseudoconvex either.

From the claim we get $p_0, p_1, \dots, p_{n-1} \in T_q$, which is impossible since $T_q \neq H$. $\square$

To study the geometry of D along H , we first consider another auxiliary domain $D_2 \supset D_1$, which may be strictly larger than D .

Lemma 2.2 *The set D_2 obtained from D_1 by adding every point $z \in H$ which has an ambient neighborhood U satisfying $U \setminus H \subset D_1$, is a pseudoconvex domain.*

Lemma 2.2 is a consequence of a classical result of Grauert and Remmert [1, Satz 4], since D_2 is pseudoconvex near points on $\partial D_2 \setminus H$ and $\partial D_2 \cap H$ is a thin set without removable points (in the terminology of [1]). For the reader's convenience, we provide a short proof using Hörmander's L^2 theory [3] and an idea from [8]. The final version of the argument was communicated by P. Pflug.

Proof of Lemma 2.2 Let $G \subset \mathbb{C}^n$ be a domain. We call a pair (U, V) of domains in $\mathbb{C}^n$ an *extension pair* for G if **(i)** $V \not\subset G$, **(ii)** $V \cap G \neq \emptyset$, **(iii)** U is a connected component of $V \cap G$, and **(iv)** for every $f \in \mathcal{O}(G)$, there is $\tilde{f} \in \mathcal{O}(V)$ which coincides with f on U . By classical Cartan-Thullen theory (see [2, 4]), G is pseudoconvex if and only if it does not admit any extension pair.

Obviously D_2 is a domain. Let us assume that D_2 is not pseudoconvex, implying the existence of an extension pair (U, V) for D_2 . Note that $(U \setminus H, V \setminus H)$ satisfies the properties **(i–iii)** for $G = D_1$. Indeed, **(ii)** and **(iii)** hold since we have subtracted a subspace of real codimension 2. As for **(i)**, $(V \setminus H) \subset D_1$ would imply $V \cap H \subset D_2$ (by the definition of D_2) and yield the contradiction $V \subset D_2$.

We will exploit a structure theorem on holomorphic functions with polynomial growth. Let $\delta_{D_1}(z)$ be the euclidean distance between $z \in D_1$ and ∂D_1 . By [4, Proposition 2.5.5] and Lemma 2.1, D_1 is also a domain of holomorphy with respect to the Banach space

$$\mathcal{O}^{(2n+1)}(D_1) = \left\{ f \in \mathcal{O}(X) \mid \delta_{D_1}^{(2n+1)} f \text{ is bounded on } D_1 \right\},$$

meaning that for every point in $w \in \partial D_1$, there is a function in $\mathcal{O}^{(2n+1)}(D_1)$ without holomorphic extension across w .

Since $(U \setminus H) \subsetneq (V \setminus H)$ there is a point $z_0 \in \partial(U \setminus H) \cap (V \setminus H)$. From **(iii)** we get $z_0 \in \partial D_1$. The structure theorem yields $f \in \mathcal{O}^{(2n+1)}(D_1)$ without extension through z_0 . Riemann's theorem on removable singularities implies that $g = z^{2n+1} f$ is holomorphic on D_2 and therefore $g|_U$ admits an extension $\tilde{g} \in \mathcal{O}(V)$. But this means that $f|_U$ has an extension $\tilde{g}/z^{2n+1} \in \mathcal{O}(V \setminus H)$, a contradiction to the choice of z_0 and f . The proof of Lemma 2.2 is complete. $\square$

By construction we have $D \subset D_2$ and $D \setminus H = D_1 = D_2 \setminus H$. We continue by studying $\widehat{D} = D \cap H$, $\widehat{D}_2 = D_2 \cap H$. These are of course open subsets of H satisfying $\widehat{D} \subset \widehat{D}_2$.

Lemma 2.3 *The region $\widehat{D}$ is a union of connected components of $\widehat{D}_2$.*

Proof of Lemma 2.3 The proof relies on a propagation property. Assume that the statement fails, i.e. that there is a connected component W of $\widehat{D}_2$ with $\emptyset \neq W \cap \widehat{D} \subsetneq W$. Using that connected open sets are path connected, we find a continuous curve $\gamma : [0, 1] \rightarrow W$ satisfying $\gamma(0) \in \widehat{D}_2 \setminus \widehat{D}$ and $\gamma((0, 1]) \subset \widehat{D}$. The crucial step is to replace γ by a well-chosen straight segment to which we can apply propagation arguments. We claim:

There is a compact straight real segment $I \subset H$ contained in a complex line through one of the points $p_0, p_1, \dots, p_{n-1}$ such that one extremal point of I is contained in $\widehat{D}_2 \setminus \widehat{D}$ and all other points in $\widehat{D}$.

In preparation we construct a suitable parametrization of H near $w = \gamma(0)$. For two points p and q we denote by $\overrightarrow{pq}$ the vector pointing from p to q and by $[p, q]$ the straight (real) segment linking the points. Observe that there are $n - 1$ $\mathbb{C}$ -independent vectors among $v_j = \overrightarrow{wp_j}$, $j = 0, \dots, n - 1$. In fact, the v_j span H , and we can select a basis.

Suppose $v_1, \dots, v_{n-1}$ independent. Set

$$F_1(\zeta_1) = w + \zeta_1 v_1,$$

and define inductively for $k = 2, \dots, n - 1$ the mappings

$$F_k(\zeta_1, \dots, \zeta_k) = F_{k-1}(\zeta_1, \dots, \zeta_{k-1}) + \zeta_k \overrightarrow{F_{k-1}(\zeta_1, \dots, \zeta_{k-1}) p_k}.$$

Taking the differential at $\zeta = 0$, we see that the mapping $F = F_{n-1}$ is a biholomorphism of the $(n - 1)$ -dimensional polydisc $\{\max |\zeta_j| < \eta\}$ onto a neighborhood of w in H , if $\eta > 0$ is small. It is crucial that for $\zeta_1, \dots, \zeta_\ell$ fixed, the image of

$$\lambda \mapsto F(\zeta_1, \dots, \zeta_\ell, \lambda, 0, \dots, 0)$$

is contained in a line through p_ℓ .

After shortening γ , we may assume that γ is contained in

$$U = F(\{\max |\zeta_j| < \eta\}) \subset W.$$

We link $w = \gamma(0)$ and $\gamma(1) = F(\hat{\zeta}_1, \dots, \hat{\zeta}_{n-1})$ by the concatenation of the straight segments

$$[w, F(\hat{\zeta}_1, 0, \dots, 0)], [F(\hat{\zeta}_1, 0, \dots, 0), F(\hat{\zeta}_1, \hat{\zeta}_2, 0, \dots, 0)], \dots, \\ [F(\hat{\zeta}_1, \dots, \hat{\zeta}_{n-2}, 0), F(\hat{\zeta}_1, \dots, \hat{\zeta}_{n-1})].$$

By construction each of these segments lies on a complex line through one of the p_j . Since there is a last point in $\widehat{D}_2 \setminus \widehat{D}$ on the concatenation, we obtain a segment I as required by taking an appropriate subsegment. The claim is proved.

By construction I lies on a complex line L passing through one of the points $p_0, \dots, p_{n-1}$. Hence the intersection of D with the plane

$$Q = \{(z_1, \dots, z_n) \in \mathbb{C}^n : (z_1, \dots, z_{n-1}, 0) \in L\}$$

with D is pseudoconvex. On the other hand, the one-parameter family of flat complex discs

$$\Delta_{(z_1, \dots, z_{n-1})} = \{(z_1, \dots, z_{n-1}) \times \{|z_n| \leq \varepsilon\}, (z_1, \dots, z_{n-1}, 0) \in I,$$

violates the continuity principle (with respect to $D \cap Q$) if $\varepsilon > 0$ is small enough. This contradiction completes the proof of Lemma 2.3. $\square$

Since D is obtained by taking out of D_2 a collection of properly embedded complex hypersurfaces, it is pseudoconvex. The proof of Theorem 1.1 is complete. $\square$

The main theorem and its proof allow a natural generalization to Riemann domains $\pi : X \rightarrow \mathbb{C}^n$, $n \geq 3$. For an affine complex 2-plane P we consider $P_X = \pi^{-1}(P)$. For a nonpseudoconvex X , we call $z \in \mathbb{C}^n$ *exceptional* if the intersection P_X is pseudoconvex for every 2-plane P passing through z , and denote by E the set of exceptional points. We obtain

Corollary 2.4 *If $\pi : X \rightarrow \mathbb{C}^n$, $n \geq 3$, is not pseudoconvex, then E is contained in a complex affine subspace of $\mathbb{C}^n$ of codimension two.*

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