Regularity and uniqueness-related properties of solutions with respect to locally integrable structures

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Abstract

We prove that a smooth generic embedded CR submanifold of $\mathbb{C}^n$ obeys the maximum principle for continuous CR functions if and only if it is weakly 1-concave. The proof of the maximum principle in the original manuscript has later been generalized to embedded weakly q-concave CR submanifolds of certain complex manifolds.

We give a generalization of a known result regarding automatic smoothness of solutions to the homogeneous problem for the tangential CR vector fields given local holomorphic extension. This generalization ensures that a given locally integrable structure is hypocomplex at the origin if and only if it does not allow solutions near the origin which cannot be represented by a smooth function near the origin.

We give a sufficient condition under which it holds true that if a smooth CR function f on a smooth generic embedded CR submanifold $M \subset \mathbb{C}^n$, vanishes to infinite order along a C^∞-smooth curve $\gamma \subset M$ then f vanishes on an M-neighborhood of γ.

We prove a local maximum principle for certain locally integrable structures.

Keywords: Maximum principle, hypocomplexity, locally integrable structure, hypoanalytic structure, weak pseudoconcavity, uniqueness, CR functions

Sammandrag

Vi visar att för släta generiska inbäddade CR-mångfalder i $\mathbb{C}^n$ gäller att maximumprincipen för kontinuerliga CR-funktioner håller om och endast om CR-mångfalden är svagt 1-konkav. Beviset av satsen i det ursprungliga manuset har senare generaliserats till svagt q-konkava CR-delmångfalder av vissa komplexa mångfalder.

Vi generaliserar en känd sats om automatisk släthet för lösningar till de tangentiella CR-ekvationerna, givet existensen av lokal holomorf utvidgning. Generaliseringen ger att en lokalt integrerbar struktur är hypokomplex i origo om och endast om den inte tillåter icke-släta lösningar nära origo.

Vi bevisar att en ifall en slät CR function f på en slät generiskt inbäddad CR mångfald $M \subset \mathbb{C}^n$ försvinner till oändlig ordning längs en reell slät kurva $\gamma \subset M$ sådan att f uppfyller vissa ytterligare tillväxtvillkor, så måste f försvinna på en omgivning av γ in M.

Vi bevisar en lokal maximumprincip för vissa lokalt integrerbara strukturer.

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Contents

List of Papers

This thesis is mainly based on the following papers, herein referred by their Roman numerals:

I Egmont Porten & Abtin Daghighi
The maximum principle for continuous CR functions on weakly pseudoconcave CR manifolds. Manuscript (2012)

II Egmont Porten & Abtin Daghighi
On the relation between regularity and hypocomplexity. Manuscript (2012)

III Abtin Daghighi
On a uniqueness condition for CR functions on hypersurfaces. Manuscript (2013)

IV Abtin Daghighi
A maximum principle for locally integrable structures. Manuscript (2013)

Chapter 1

Introduction

This thesis concerns uniqueness-related properties of solutions with respect to locally integrable structures underlying hypoanalytic structures, mainly structures defining CR submanifolds of $\mathbb{C}^n$. In each result we have generalized properties which are known for holomorphic functions on complex manifolds (and sometimes also known for CR distributions). Section 3 treats a generalization, to locally integrable structures, of a known recent result on the regularity of solutions and the relation to hypocomplexity. Section 2 treats the maximum principle for continuous CR functions on weakly pseudoconcave CR manifolds. Section 4 concerns a special set of circumstances when unique continuation follows from vanishing to infinite order and some additional growth properties. Section 5 gives a local maximum principle for certain locally integrable structures. Before we present the results of the papers we recall here some basics on ellipticity, uniqueness and the maximum principle, and mention in which paper they are relevant.

When it comes to uniqueness results, one of the main tools in partial differential equations is different kinds of maximum principles (this is, e.g., the standard way of proving uniqueness for the Dirichlet problem on bounded domains of $\mathbb{R}^n$). Recall that one version of the maximum principle for holomorphic functions is that given a bounded domain $U \subset \mathbb{C}^n$, a holomorphic function f on U which is continuous up to the boundary, must satisfy $\max_{z\in\overline{U}} |f(z)| = \max_{z\in\partial U} |f(z)|$. We consider in Section 2 the generalization to CR functions in the sense that the maximum principle is said to hold for continuous CR functions on a CR manifold M if given a domain $U \Subset M$, a continuous CR function f on U which is continuous up to the boundary, must satisfy $\max_{z\in\overline{U}} |f(z)| = \max_{z\in\partial U} |f(z)|$. For the basic definitions on CR manifolds see Appendix A.1. We prove a maximum principle for restrictions of holomorphic polynomials to smooth weakly 1-concave CR submanifolds of $\mathbb{C}^n$ (see Section 2). Local convexification and local approximation by entire functions are the essential tools used in the proof. The result of the original manuscript has been generalized to weakly q-concave CR submanifolds of certain complex manifolds.

Section 3 requires some knowledge on locally integrable structures and hypoanalytic structures. The interested reader will find some basic definitions on hypoanalytic structures in Appendix A.2. It is known that weakly (L^1_{loc}) holomorphic functions are necessarily holomorphic (in particular smooth), see, e.g., Krantz [46], p.200.

Definition 1.0.1 (See e.g. Treves [74], p.19)**.** Let $P = \sum_{|\alpha|\leq q} \phi_\alpha(x)\partial^\alpha$ be a linear partial differential operator of order q with smooth coefficients on a domain $\Omega \subset \mathbb{R}^N$. The operator P is called elliptic if the symbol $\sum_{|\alpha|=q} \phi_\alpha(x)\xi^\alpha \neq 0$, for all $x \in \Omega$ and all $\xi \neq 0$, $\xi \in \mathbb{R}^N$. The operator P is called *hypoelliptic* if for any open $\omega \subset \Omega$, $Pu \in C^\infty(\omega)$ implies $u \in C^\infty(\omega)$.

The notion has been generalized to CR manifolds by Nacincovich & Porten [53], and in Section 3 we treat the analogue notion for locally integrable structures. That a partial differential equation has only smooth solutions is a strong condition and nonetheless the solutions form rich function spaces already in the case of holomorphic functions. The theory of CR geometry and that of hypoanalytic structures do not always occupy the same readers and therefore results of the kind we present in Section 3 may help to combine the theories. Following this same theme, Section 5 gives a local maximum principle for locally integrable structures. In Section 4 we shall consider the fact that, continuation of a CR function to a larger domain may or may not be unique depending on the circumstances, e.g., the geometric properties of the CR manifold it is defined on. Assume for simplicity that given a manifold, M, and a linear local system of equations, it is possible to specify a submanifold, N, such that whenever a solution vanishes on N, it automatically vanishes identically on a neighborhood, $\tilde{N}$, of N in M. Clearly the difference of two solutions which agree on N will have an extension which is identically zero on $\tilde{N} \subset M$, whence the continuation from N to $\tilde{N}$ of any solution will be *unique*. As an example, consider the case of holomorphic functions on complex manifolds (i.e., homogeneous solutions to the Cauchy–Riemann equations). Due to the identity principle it suffices that a holomorphic function on a given complex manifold, vanish on a open subset, for the function to vanish identically. Also (due to the property of being complex analytic) it suffices that the infinite jet vanishes at a single point, for the function to vanish identically near that point.

We are in Section 4, interested in sufficient conditions for a submanifold $N \subset M$, for local unique continuation to hold true for the subclass of smooth (i.e., C^∞) CR functions (we write CR^k for C^k-smooth CR functions). The result of Section 4 involves minimality for CR submanifolds and local CR orbits. For a smooth vector field X on an open $\Omega \subset \mathbb{R}^n$ and any point $p \in \Omega$ there exists a a unique integral curve, κ, satisfying $\kappa : [0, T] \to \Omega$, (for a maximal T) $\dot{\kappa}(t) = X(\kappa)(t)$, of X, which passes through p when $t = 0$ i.e. $\kappa(0) = p$ (see e.g. Jost [44], p.52). We shall denote this integral curve by $t \mapsto X_t(p)$. Moreover, it a classic result that existence, uniqueness and smoothness with respect to parameters for the differential equation which defines an integral curve holds true in the following sense (where X is allowed to depend upon a parameter ϑ, $X =: X_\vartheta$).

Theorem 1.0.2 (See e.g. Hartmann [32], p.94)**.** *Let $\eta(t, \gamma, \vartheta)$ be continuous on an open (t, y, ϑ)-set, E, with the property that for every $(t_0, \gamma_0, \vartheta) \in E$, the initial value problem, $\gamma'(t) = \eta(t, \gamma, \vartheta)$, with ϑ fixed, has a unique solution, depending on $(t, t_0, \gamma_0, \vartheta)$, defined for a maximal interval $t \in (a, b)$. Then a, b depend on t_0, γ_0, ϑ, $t \in (a(t_0, \gamma_0, \xi), b(t_0, \gamma_0, \vartheta))$, such that $a(t_0, \gamma_0, \xi)$ (or $b(t_0, \gamma_0, \xi)$) is a lower (upper) semicontinuous function of $(t_0, \gamma_0, \vartheta) \in E$ and the solution depending on $(t, t_0, \gamma_0, \vartheta)$ is continuous on the set $a \le t \le b$, $(t_0, \gamma_0, \vartheta) \in E$.*

We shall mainly be interested in the case when $f(t, \gamma, \vartheta) = (X_\vartheta \gamma)(t)$, where X_ϑ is a vector field. Let H be a collection of smooth vector fields on Ω. By a *polygonal path of a finite number of integral curves, of vector fields in H* joining $q' \in \Omega$ to $q \in \Omega$ we mean a piecewise smooth curve $\kappa : [0, 1] \to \Omega$ such that $\kappa(0) = q$, $\kappa(1) = q'$ and $0 = s_0 < s_1 < \cdots < s_k = 1$ (for some positive integer R) such that

$$\kappa(s) = X^j_{t_j(s)}(\kappa(s_{j-1})), \quad s_{j-1} \le s \le s_j, \quad 1 \le j \le k, \tag{1.1}$$

where $X^j \in H$ and $t_j(s)$ is a smooth diffeomorphism of $[s_{j-1}, s_j]$ onto some closed interval of $\mathbb{R}$ with $t_j(s_{j-1}) = 0$.

Definition 1.0.3 (See Baouendi et al. [4], p.94)**.** Let M be a smooth CR manifold and let $p \in M$. By a known theorem (see Baouendi et al. [4], p.68) there exists a C^∞-smooth submanifold $W \subset M$, $p \in W$, satisfying (i) if $W' \ni p$ is another C^∞-smooth submanifold to which all vector fields of T^cM are tangent at every point then there is an open $V \subset M$ with $W \cap V \subset W' \cap V$. (ii) For every open $U \subset M$, $p \in U$, there exists $N \in \mathbb{Z}_+$, and open $V_1 \subset V_2 \subset U$, with $p \in V_1$, such that any $q \in V_1 \cap W$ can be reached by a polygonal path of N integral curves, of vector fields in T^cM, contained in $W \cap V_2$.

We denote by $\mathfrak{o}(p)$, the members of the germ[1] of W at p, such that the tangent space at each point of the member contains $T^c_q M$. We call $\mathfrak{o}(p)$ the *local* CR*-orbit at* p.

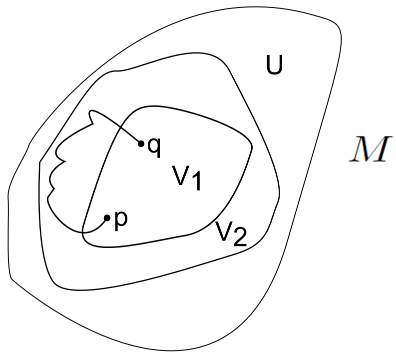

Figure 1.1: Explanatory figure for the condition (ii) of Definition 1.0.3. In this case $N = 4$.

A continuous function is a CR function on M if and only if $f|_{\mathfrak{o}(p)}$ is a CR function for every (member of every) orbit (see Jöricke [43], p.561).

Definition 1.0.4 (See Baouendi et al. [4], p.20)**.** We say that M is *minimal* at p_0 if there does not exist any real submanifold, S, of M passing through p such that $T^c_p M$ is tangent to S at every $p \in S$, but $\dim_{\mathbb{R}} S < \dim_{\mathbb{R}} M$.

Minimality at p is in some literature called as Tumanov's minimality condition is satisfied at p.

[1]By a germ of a manifold M at p we mean an equivalence class on the family of connected submanifolds of M passing through p under the relation defined as follows: For two submanifolds A and B passing through p, we define A and B to belong to the same germ at p, if there exists an open $U \subset M$, $p \in U$, such that $U \cap A = U \cap B$.

Chapter 2

About Paper I: The maximum principle for continuous CR functions on weakly pseudoconcave CR manifolds

Paper I contains a generalization of the maximum principle for holomorphic functions to the case of continuous CR functions on smooth CR submanifolds (for simplicity we sometimes denote the set of C^k-smooth CR functions by CR^k, e.g., the set of continuous CR functions is sometimes denoted CR^0) without strictly pseudoconvex points. It is clear that if a real C^2-smooth hypersurface $M \subset \mathbb{C}^n$ is strictly convex near $p_0 \in M$, regarded as a subset of $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ and if $z_j = x_j + iy_j$, $j = 1, \ldots, n$ are holomorphic coordinates for $\mathbb{C}^n$ (such that (x, y) are Euclidean coordinates for $\mathbb{R}^{2n}$) then there is an open $U \subset \mathbb{C}^n$, $p_0 \in U$ and a defining function ρ with $U \cap M = \{\rho = 0\}$ such that $\sum_{j,k=1}^{n} \frac{\partial^2 \rho(p_0)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0$, for all w satisfying $\sum_{j=1}^{n} \frac{\partial \rho(p_0)}{\partial z_j} w_j = 0$. Conversely if such a local defining function exists near p_0 then it is a result of Ramanujan (see e.g. Krantz [46], p.134) that M is, near p_0, the image under a biholomorphic map, of a strictly convex hypersurface. Whence it is possible, for hypersurfaces, to use as definition of strict pseudoconvexity at p_0, the property of being, locally near p_0, the image under a biholomorphic map, of a strictly convex hypersurface.

Definition 2.0.5 (Strict pseudoconvexity)**.** Let $M \subset \mathbb{C}^n$ be a generic CR submanifold. A point $p_0 \in M$ will be called a *strictly pseudoconvex point of* M if there exists an open neighborhood U, of p_0, in M, such that U is contained in a strictly pseudoconvex hypersurface.

(An N-dimensional submanifold $M \subset \mathbb{C}^n$, is locally near any of its points, the image, $Z(\omega)$, of an embedding $Z : \omega \hookrightarrow \mathbb{C}^n$, for an open $\omega \subset \mathbb{R}^N$. We shall in Section 5, work with conditions involving the existence of certain choices of Z.) The absence of strictly pseudoconvex points is equivalent to weak 1-concavity. In order to give the definition of weak q-concavity, we first recall the definition of the vector valued Levi form. Let $M \subset \mathbb{C}^n$ be a generic CR submanifold and let $p \in M$. The Euclidean metric on $\mathbb{C} \otimes T_p\mathbb{C}^n$ induces a metric on $\mathbb{C}\otimes T_pM$. In particular the quotient space $\mathbb{C}\otimes T_pM/(H_p^{1,0}M \oplus H_p^{0,1}M)$ can be identified with the orthogonal supplement of $H_p^{1,0}M \oplus H_p^{0,1}M$ and furthermore we can identify $\mathbb{C} \otimes T_pM/(H_p^{1,0}M \oplus H_p^{0,1}M) = \mathbb{C} \otimes (T_pM/T_p^cM)$. Given $X \in H_p^{1,0}M$ we define $\mathcal{L}(X) := \frac{1}{2i}[\overline{\tilde{X}}, \tilde{X}]_p \bmod \mathbb{C} \otimes T_p^cM$, where $\tilde{X}$ is any smooth section of $H^{1,0}M$ extending X. $\mathcal{L}$ is real-valued so the image which lies in $\mathbb{C} \otimes (T_pM/T_p^cM)$, can be identified with the real vector space T_pM/T_p^cM.

Definition 2.0.6 (See, e.g., Porten [58])**.** Let $M \subset \mathbb{C}^n$ be a generic CR submanifold and let $p \in M$. Set $\chi_p := \{\xi \in T_p^*M : \xi|_{T_p^cM} \equiv 0\}$. The *directional Levi form at p in codirection* $\xi \in \chi_p$ is defined as $\mathcal{L}^\xi(X) := \langle \xi, \mathcal{L}(X)\rangle$. M is called *strictly/weakly q-concave at p* if for each $\xi \neq 0$ the Hermitian form[1] $\mathcal{L}^\xi(\cdot)$ has at least q negative/nonpositive eigenvalues.

There exists many previous results on the maximum principle for CR functions, some making explicit use of the Levi form of the CR manifold, others only implicitly. There are known analogues (to the maximum principle for holomorphic functions) for CR functions on appropriate CR manifolds. We mention the following works, Rossi [61] and [62] (textbook version can be found in Stout [70], p.78), Hill & Nacinovich [34] and [35], Ellis et al. [26] and Iordan [41] (whose works often involve so-called extreme points), Berhanu [15], Carlson & Hill [22], Sibony [69], Henkin & Michel [33] (their work primarily concern Hartog's phenomenon), and specifically the recent result of Berhanu & Wang [14].

[1]Recall that $h\colon V \times V \to \mathbb{C}$ on a complex vector space V is called a Hermitian form if it is linear in the first coordinate and such that $h(X,Y) = \overline{h(Y,X)}$. Every Hermitian form has an associated Hermitian matrix, A, such that $h(X,Y) = XA_h\overline{Y}^T$. In our situation we consider a Hermitian real-valued $\mathcal{L}(X,Y) := \frac{1}{2i}[\overline{\tilde{X}}, \tilde{Y}]_p \bmod \mathbb{C}\otimes T_p^cM$. Then we speak of the eigenvalues of the Hermitian matrix $A_{\mathcal{L}^\xi}$.

2.1 The necessity of weak pseudoconcavity

We shall construct a counterexample given a strictly pseudoconvex point on a smooth hypersurface. The necessity then follows due to the following result (see Tumanov [77], p.448): *Let $M \subset \mathbb{C}^n$ be a smooth generic CR submanifold and let $p \in M$. If M is strictly Levi pseudoconvex at p, then locally near p, M is contained in a strictly pseudoconvex hypersurface in $\mathbb{C}^n$.* For the case when $M \subset \mathbb{C}^n$ is a smooth real hypersurface, which is strictly pseudoconvex at $p \in M$, this is a well-known result (which can be found in Narasimhan [54]). Namely there exists an ambient neighborhood U of p and a biholomorphism ϕ, such that $\phi(U \cap M)$ is a strictly convex hypersurface in $\phi(U) \subset \mathbb{C}^n$ (the proof consists of writing out the Taylor expansion of the local defining function for M near p and via a biholomorphic coordinate change make sure that only the Levi form[2] remains as second order terms (see, e.g., Saracco [65] for a short proof). Denote the new local holomorphic coordinate system $(z_1, \dots, z_n)$ near p in $\mathbb{C}^n$, with $z(p) = 0$, so that M is strictly convex near the origin in these coordinates. Denote $y := \operatorname{Im} z_n$ and $x = \operatorname{Re} z_n$. We may assume that M in these coordinates has local graph representation on a sufficiently small ambient neighborhood V of origin according to $M \cap V = \{y = h(z_1, \dots, z_{n-1}, x)\}$, $h \in C^\infty(A, \mathbb{R})$, $A \subset \mathbb{C}^{n-1} \times \mathbb{R}$. In particular we can assume that $M \cap V \setminus \{0\}$ belongs to $\{y > 0\}$. Then the function $f := e^{-iz_n}$ satisfies $|f(0)| = 1$, but $\left|e^{iz_n}\right| = e^y < 1$, for each point $z \in M \cap V \setminus \{0\}$.

2.2 Result

We now state our main result.

Theorem 2.2.1 (Main theorem on the maximum principle for submanifolds of $\mathbb{C}^n$)**.** *Let M be a generic smooth weakly 1-concave CR submanifold of $\mathbb{C}^n$, and let $\Omega \Subset M$. Then,*

$$f \in CR^0(\Omega) \cap C^0(\overline{\Omega}) \quad \Rightarrow \quad \max_{z \in \overline{\Omega}} |f(z)| = \max_{z \in \partial\Omega} |f(z)| . \tag{2.1}$$

[2]In the case of a smooth hypersurface M the Levi form at $p_0 \in M$, is particularly easy to describe. Let $U \ni p_0$ be an open subset such that $M \cap U = \{\rho = 0\}$, for some $\rho : U \to \mathbb{R}$ with $|\nabla\rho(p_0)| = 1$. The Levi form at $p_0 \in M$, is (a constant multiple of), $\mathcal{L}_{M,p_0}(W) = -\left(\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p_0)\zeta_j\bar{\zeta}_k\right)\nabla\rho(p_0)$, $\quad W = \sum_{k=1}^n \zeta_k \frac{\partial}{\partial z_k} \in H^{1,0}_{p_0}M$, where $\nabla\rho(p_0)$ is often dropped (it is a vector which spans the real one dimensional $N_{p_0}M$) and the Levi form is identified with the restriction, to $H^{1,0}_{p_0}M$, of the complex Hessian of ρ, see Boggess [21], p.163, or Ayrapetyan & Khenkin [1], p.47.

The proof of the main theorem relies upon a local version of the result. In order to prove such a local version we first prove it for restrictions of holomorphic polynomials (Proposition 2.2.2) to the elements of a neighborhood basis consisting of intersections with sufficiently small ambient balls.

Proposition 2.2.2. *Let $M \subset \mathbb{C}^n$ be a generic weakly 1-pseudoconcave CR submanifold and let $p \in M$. Then for any holomorphic polynomial and sufficiently small $r > 0$ it holds true that,*

$$\max_{z \in \partial B_p(r) \cap M} |P(z)| = \max_{z \in \overline{B_p(r) \cap M}} |P(z)| . \tag{2.2}$$

By assuming the contrary for the case when $|P|^2$ is replaced by a strictly plurisubharmonic function, we find a local hypersurface which is a level set of a strictly plurisubharmonic function $\tilde{\rho}$ such that M must touch $\tilde{\rho}$ from the convex side, and deduce that M must itself have a strictly pseudoconvex point.

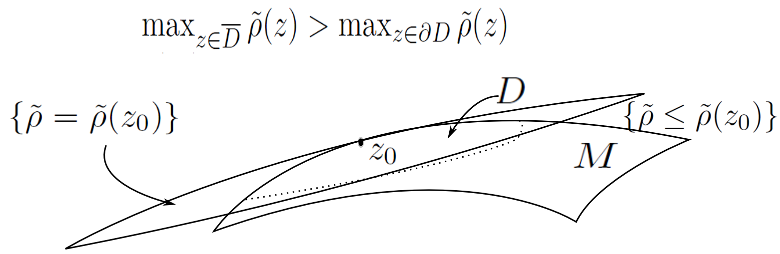

Figure 2.1: Explanatory figure for the idea of the proof of Proposition 2.2.2, when p, the plurisubharmonic function $|P|^2$ and $B_p(r) \cap M$, are replaced by z_0, a strictly plurisubharmonic function ρ, and a domain $p \in D \subset M$ respectively. Assuming eqn 2.2 fails, we can find another strictly plurisubharmonic function $\tilde{\rho}$, with the properties depicted in the figure. The main argument is that $\{\tilde{\rho} = \tilde{\rho}(z_0)\}$ must be strictly pseudoconvex at z_0 whereas M is nowhere strictly pseudoconvex.

We obtain Proposition 2.2.2 by approximating the plurisubharmonic $|P|^2$ with strictly plurisubharmonic functions. The proof of Theorem 2.2.1 is based on contradiction, namely assuming the result fails, we construct a counterexample to the local version of the result (see Figure 2.2).

The proof in the original manuscript of the maximum principle for weakly 1-concave CR submanifolds of $\mathbb{C}^n$ has later been generalized to embedded weakly q-concave C^2-smooth CR submanifolds of complex manifolds which carry a C^2-smooth real-valued function with sufficiently many positive eigenvalues of its Levi form at each point.

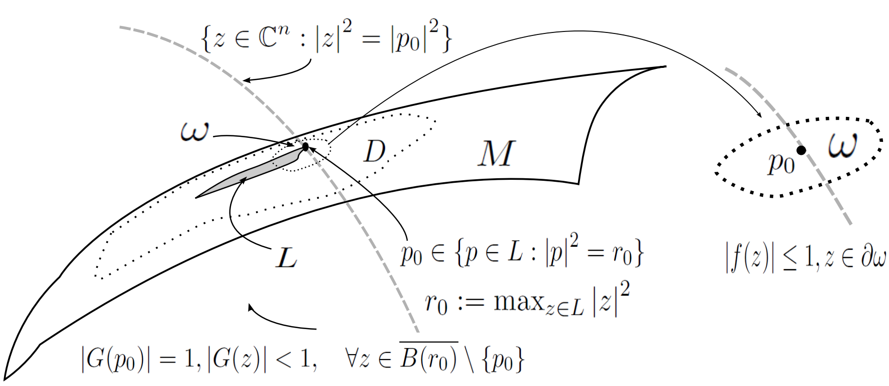

Figure 2.2: Assuming Theorem 2.2.1 fails there is a domain $D \subset M$ and $f \in CR(\overline{D})$, whose peak set L satisfies $L \Subset D$. We construct a counterexample to the local version of the result, of the form $F := f^N \cdot G$, form some appropriate holomorphic peak function G and sufficiently large integer N, in the sense that there is a domain $p_0 \in \omega \subset M$ such that $F|_{\overline{\omega}}$ does not attain maximum on the boundary.

Chapter 3

About paper II: A generalization to locally integrable structures of the relation between regularity and hypocomplexity

Recently, Nacinovich & Porten [53] proved that local ambient holomorphic extension of CR distributions near a point is equivalent to the property that every CR distribution near that point is representable by a smooth function near the point. In Paper II we consider whether such results can be generalized to locally integrable structures. First we cite the original result (the definitions required for the statement can be found in Appendix A.1.4).

Theorem 3.0.3 (Nacinovich & Porten [53])**.** *Let M be a CR manifold, locally CR-embeddable at $p_0 \in M$. Then M has the holomorphic extension property at p_0 if and only if M is CR-hypoelliptic at p_0.*

Let Ω be a smooth real manifold. Given a complex locally integrable subbundle $L \subset \mathbb{C} \otimes T\Omega$ (see Definition A.2.2) we define $T' = L^{\perp}$ in $\mathbb{C} \otimes T^*\Omega$. A hypoanalytic structure determines a unique locally integrable structure, namely the cotangent structure bundle T' spanned by the differentials of the components of the hypoanalytic chart maps. However, the choice of hypoanalytic chart is not necessarily unique, i.e., different hypoanalytic structures can yield the same locally integrable structure.

Example 3.0.4. A complex-valued function $Z \in C^\infty(\mathbb{R})$ with $dZ \neq 0$ everywhere defines a hypoanalytic structure on $\mathbb{R}$ with the single hypoanalytic chart $(\mathbb{R}, Z)$, where dZ spans $T' = \mathbb{C} \otimes T^*\mathbb{R}$. If $\hat{Z}$ is another C^∞-smooth function on $\mathbb{R}$ with $d\hat{Z} \neq 0$ everywhere then Z and $\hat{Z}$ define the same hypoanalytic structure if and only if, locally they are holomorphic functions of each other. If we choose Z as a real-analytic function (e.g. $Z(x) = x$) and $\hat{Z}$ as a non-real-analytic function, then they cannot be holomorphic functions of each other. (see Baouendi et al. [5], p.335 or Treves [72], p.126).

Definition 3.0.5. Let Ω be a real smooth manifold equipped with a locally integrable structure L. The pair (Ω, L) is called *hypocomplex at* p if there is an open neighborhood $U \subset \Omega$ of p and a local smooth chart $Z = (Z_1, \ldots, Z_n)\colon U \to \mathbb{C}^n$, whose components are solutions such that for any distribution u on a neighborhood $U' \subset \Omega$ of p there is a holomorphic function $\tilde{u}$ defined on a neighborhood of $Z(p)$ such that $u = \tilde{u} \circ Z$ on a neighborhood of p in U'.

The proof of the main theorem in this section uses a generalization of Tumanov's wedge extension theorem to the setting of hypoanalytic structures, due to Marson [49]. It is therefore justified to give the not so common definitions of wedge and wedge extension in the context of locally integrable structures here, before we cite Marson's result (Theorem 3.0.10). Let $\Omega \subset \mathbb{R}^{n+l}$, $0 \in \Omega$ be an open subset and let $L \subset \mathbb{C} \otimes T\Omega$ be a C^∞-smooth complex subbundle of rank l which is integrable at 0 in the sense that there is a neighborhood $\Omega_0 \ni 0$ and smooth functions $Z_1, \ldots, Z_n$, defined on Ω_0, with $\mathbb{C}$-linearly independent differentials on Ω_0, with $XZ_j = 0$, for all smooth sections X of L. That (Ω_0, L, Z) defines a hypoanalytic structure (with a single chart) implies that there are local C^∞-smooth coordinates $(x_1, \ldots, x_r, s_1, \ldots, s_{n-r}, y_1, \ldots, y_l)$ for Ω centered at the origin, and a smooth ϕ such that,

$$Z_j(x, s, y) = x_j + iy_j, \qquad 1 \leq j \leq r, \tag{3.1}$$

$$Z_{r+j}(x, s, y) = s_j + i\phi_j(x, s, y), \qquad 1 \leq j \leq n - r, \tag{3.2}$$

with $\phi(0) = d\phi(0) = 0$, see for example Treves [72], p.39. Note that this implies $r \leq l$.

Definition 3.0.6 (Wedge). Let $\Omega \subset \mathbb{R}^{n+l}$ be an open subset, $0 \in \Omega$, and let $L \subset \mathbb{C} \otimes T\Omega$ be a locally integrable structure on Ω of rank l. Let Z be a hypoanalytic chart on an open neighborhood $\omega \subseteq \Omega$ of the origin and let $(z, w) \in \mathbb{C}^r \times \mathbb{C}^{n-r}$ denote complex coordinates such that Z has the representation given by eqn 3.1 and eqn 3.2 with $z_j = x_j + iy_j$, $1 \leq j \leq r$, and $s_j = \operatorname{Re} w_j$, $1 \leq j \leq n-r$. Let $0 \in U' \subset \mathbb{R}^{l-r}$ and $0 \in V \subset \mathbb{C}^n$, where U', V are open subsets such that $\{(\operatorname{Re} z, \operatorname{Re} w, \operatorname{Im} z, y') : y' \in U', (z, w) \in V\} \subset \omega$. Let $\Gamma \subset \mathbb{R}^{n-r}$ be a strictly convex open cone. A *wedge at* $0 \in \Omega$ is an open set in $\mathbb{C}^n$ of the form $\mathcal{W}(U' \times V, \Gamma) = \{(z, w) \in V : \exists y' \in U' \text{ such that } \operatorname{Im} w - \phi(\operatorname{Re} z, \operatorname{Re} w, \operatorname{Im} z, y') \in \Gamma\}$, (where ϕ is given by eqn 3.2).

Let Ω be a smooth real manifold and let ω be an open neighborhood of 0 in Ω. Given a set, $\mathcal{F}$, of locally defined smooth vector fields on Ω we shall say that a point $q \in \omega$ can be reached from 0 by the concatenation of integral curves of the members of $\mathcal{F}$ if (here we follow the definition found in Berhanu et al. [16], p.101) there exists $\gamma\colon [0, c] \to \Omega$, (for a positive real number c) such that: (i) $\gamma(0) = 0, \gamma(c) = q$, (ii) there exist, $0 < t_1 < \cdots < t_n = c$, and vector fields $X_j \in \mathcal{F}$, such that $\gamma|_{[t_{j-1}, t_j]}$ is an integral curve of X_j or $-X_j$ (and the curve is contained in ω).

Definition 3.0.7 (Minimality, see e.g. Berhanu et al. [16], p.121)**.** A subbundle $L \subset \mathbb{C} \otimes T\Omega$ is called *minimal at* 0 if given an open $\omega \subset \Omega$, with $0 \in \omega$, there exists an open $\omega' \subset \omega$, $0 \in \omega'$, such that each point in ω' can be reached from 0 by a finite concatenation of integral curves of sections of $\operatorname{Re} L$ contained in ω.

Definition 3.0.8. Let $\Omega \subset \mathbb{R}^{n+l}$ be an open subset, $0 \in \Omega$, let L be a locally integrable structure on Ω of rank l and let Z be a local hypoanalytic chart on an open subset $\omega \subset \Omega$, $0 \in \omega$. Let $(z, w) \in \mathbb{C}^r \times \mathbb{C}^{n-r}$ denote holomorphic coordinates such that Z has the representation given by eqn 3.1 and eqn 3.2, for $z_j = x_j + iy_j$, $1 \leq j \leq r$, and $s_j = \operatorname{Re} w_j$, $1 \leq j \leq n-r$. The triple (ω, L, Z) is called *wedge extendible at* 0 if the following holds true: There exists open subsets $U' \subset \mathbb{R}^{l-r}$, $V \subset \mathbb{C}^n$, $0 \in U'$, $0 \in V$, such that $\{(\operatorname{Re} z, \operatorname{Re} w, \operatorname{Im} z, y') : y' \in U', (z, w) \in V\} \subset \omega$, together with a strictly convex open cone $\Gamma \subset \mathbb{R}^{n-r}$ such that for any f which is a C^1-solution defined on $Z(\omega)$ there exists a function F holomorphic on $\mathcal{W}(U' \times V, \Gamma)$ and continuous in $\mathcal{W}(U' \times V, \Gamma) \cup \{(z, w) \in V : \exists y' \in U' \text{ such that } \operatorname{Im} w = \phi(\operatorname{Re} z, \operatorname{Re} w, \operatorname{Im} z, y')\}$ (where ϕ is defined by eqn 3.2) such that $F = f$ on $\{(z, w) \in V : \exists y' \in U' \text{ such that } \operatorname{Im} w - \phi(\operatorname{Re} z, \operatorname{Re} w, \operatorname{Im} z, y') = 0\}$.

Definition 3.0.9 (Wedge extendible)**.** The pair (Ω, L), where $\Omega \ni 0$ is an open subset of a Euclidean real manifold, with a locally integrable structure $L \subset \mathbb{C} \otimes T\Omega$ is called *wedge extendible at* 0 if for any sufficiently small open neighborhood ω of 0 and any local hypoanalytic chart Z on ω, (ω, L, Z) is wedge extendible at the origin.

It is known that minimality is necessary for holomorphic extension of CR functions from generic CR submanifolds, see Baouendi & Rothschild [7]. This has been generalized as follows.

Theorem 3.0.10 (Marson [49], p.580)**.** *Let $\Omega \subset \mathbb{R}^N$ (some positive integer N) be an open subset and let $0 \in \Omega$. A locally integrable structure (Ω, L), $L \subset \mathbb{C} \otimes T\Omega$, is wedge extendible at 0 if and only if L is minimal at 0.*

In the proof we also use the fact that any two distributions $g_1 \in \mathcal{D}'(\mathbb{R}^{n_1})$, $g_2 \in \mathcal{D}'(\mathbb{R}^{n_2})$, define a unique distribution $(g_1 \otimes g_2) \in \mathcal{D}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, see, e.g., Hörmander [40], Theorem 5.1.1. A consequence is the following.

Theorem 3.0.11 (See, e.g., Friedlander & Joshi [30], p.47)**.** *If $u \in \mathcal{D}'(\mathbb{R}^n)$ then $\partial_n u = 0$ if and only if $u = v(x') \otimes 1(x_n)$ where $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, $v \in \mathcal{D}'(\mathbb{R}^{n-1})$ and $1(t)$ is the constant function, equal to unity, on $\mathbb{R}$.*

Now iteration of Theorem 3.0.11 yields that on a product space $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with coordinates (ζ, ν) a distribution $g \in \mathcal{D}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ which for fixed $\zeta \in \mathbb{R}^{n_1}$ is constant with respect to the ν-variable (in particular $\partial_\nu g$ is the zero distribution), has unique representation in terms of $g = g_1(\zeta) \otimes 1(\nu)$ where $g_1 \in \mathcal{D}'(\mathbb{R}^{n_1})$ and $1(\nu)$ is the constant function, equal to unity, on $\mathbb{R}^{n_2}$.

3.1 Result

We now give the main result. The proof is an adaptation of the proof of Theorem 3.0.3.

Theorem 3.1.1 (Main result on hypocomplexity)**.** *Let Ω be an open subset of a real Euclidean manifold and let $L \subset \mathbb{C} \otimes T\Omega$ be a locally integrable subbundle. Then (Ω, L) is hypocomplex at 0 if and only if (Ω, L) does not support solutions near 0 which cannot be represented by a smooth function near 0.*

Clearly for a generic CR submanifold failure of local ambient holomorphic extension near a point implies that the locally integrable structure which is induced by the CR structure, is not hypocomplex at that point. Therefore Theorem 3.1.1 implies that there must then exist a CR-distribution singular at the origin in that case. Typical examples are smooth CR submanifolds near non-minimal points.

We will now give an example of a hypoanalytic structure which does not have the wanted property, but in order to do this we shall first need a theorem on so-called tube manifolds which we shall use when giving the example.

Definition 3.1.2. Let $\zeta = (\zeta_1, \ldots, \zeta_N)$ denote the variable in $\mathbb{R}^N$ and $t = (t_1, \ldots, t_l)$ the variable in $\mathbb{R}^l$, for some positive integers l and N. For an open subset $U \subset \mathbb{R}^l$ and a Lipschitz mapping $\phi\colon U \to \mathbb{R}^n$ we call a manifold defined by $z = \zeta + i\phi(t)$, $z \in \mathbb{C}^N$, a *tube manifold* in $\mathbb{C}^N$.

Recall that given an open subset $\Omega \subset \mathbb{R}^{n+l}$ and a hypoanalytic structure (Ω, L, Z), where $Z\colon \Omega \to \mathbb{C}^n$ is a single hypoanalytic chart near the origin and $L \subset \mathbb{C} \otimes T\Omega$ a locally integrable subbundle of rank l, we can find coordinates $(x, s, t) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \times \mathbb{R}^l$ near 0 such that Z has the representation,

$$Z_j(x, s, t) = x_j + it_j, \qquad 1 \leq j \leq r, \tag{3.3}$$

$$Z_{r+j}(x, s, t) = s_j + i\phi_j(t), \qquad 1 \leq j \leq n - r, \tag{3.4}$$

for a smooth map ϕ satisfying $\phi(0) = 0, d\phi(0) = 0$. Assume the map $\psi(t) = (t_1, \ldots, t_r, \phi_1(t), \ldots, \phi_{n-r}(t))$ is real analytic.

Theorem 3.1.3 (Baouendi & Treves [19]). *Let ϕ and ψ be defined as above. Assume ϕ is real-analytic. Then every distribution solution f defined on Ω extends to a holomorphic function on a full neighborhood of the origin in $\mathbb{C}^n$ if and only if for every $\xi \in \mathbb{R}^n \setminus \{0\}$, the function $t \mapsto \psi(t) \cdot \xi$ does not have a local extremum for $t = 0$.*

Example 3.1.4. Let $\Omega \subset \mathbb{R}^3$ be an open subset. We define a tube manifold using $r = 0, n = 2$ and $\phi_1(t) = t^3$, $\phi_2(t) = t^4 \sin t$, in eqn 3.3 and eqn 3.4, i.e., we set,

$$Z_1(s_1, s_2, t) = s_1 + it^3, \tag{3.5}$$

$$Z_2(s_1, s_2, t) = s_2 + it^4 \sin t, \tag{3.6}$$

where (s, t) are coordinates for Ω centered at the origin. Obviously $\phi(0) = d\phi(0) = 0$. Note that Z is defined on all of Ω and dZ_1, dZ_2 are $\mathbb{C}$-linearly independent. Fix $\xi^0 \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Then $\psi(t) \cdot \xi^0 = \xi_1^0 t^3 + \xi_2^0 t^4 \sin t$ is an odd function in the variable t (independent of the choice of ξ^0). Thus $t = 0$ is not a local extremum for the (real analytic) $t \mapsto \psi(t) \cdot \xi$ for any $\xi \in \mathbb{R}^2 \setminus \{(0, 0)\}$. By Theorem 3.1.3 (Ω, L, Z) does not define a structure with respect to which all distribution solutions near 0 can be written as a composition of a holomorphic function with Z, near 0. By Theorem 3.1.1 there exists a solution (with respect to L) near 0 which is singular at 0.

Remark 3.1.5. Note that for a hypoanalytic manifold M which is also a CR submanifold of some $\mathbb{C}^n$, such that M is not hypocomplex at the origin, there are CR functions which are not the restriction of a holomorphic function. Starting from the complexification of the holomorphic tangent bundle, $L = \mathbb{C} \otimes T^c M \subset \mathbb{C} \otimes TM$, one obtains a generalization by replacing L with a more general locally integrable structure. The generalization of the CR functions thus becomes simply the solutions to the system induced by L.

3.2 A note on the proof and a corollary

If hypocomplexity at 0 holds true the generalization of CR-hypoellipticity must also hold so we only need to prove the converse. If 0 is a nonminimal point of (Ω, L), where Ω is an open subset of Euclidean manifold and L a locally integrable structure on Ω, we have the following lemma.

Lemma 3.2.1. *If 0 is non-minimal for (Ω, L) then there exists a solution on a neighborhood of 0 which cannot be represented by a smooth function near 0.*

This follows from the existence of a distribution solution u on some open neighborhood U of the origin such that the support of u is precisely the intersection of U with (a member of the germ which defines) the local $\mathrm{Re}\, L$-orbit at 0 (see Baouendi et al. p.68, for the definition of the local $\mathrm{Re}\, L$-orbit). For a proof of the later result see Treves [72], p.93. (We mention that for the CR case this result was proved by Baouendi & Rothschild [7] where the proof involves the construction of a CR-distribution u with support belonging to the local CR orbit). Thus by Lemma 3.2.1 we can assume that 0 is a minimal point of Ω. We then show that if minimality at 0 holds true and hypocomplexity at 0 fails for (Ω, L), then we can find a solution distribution which cannot be represented by a smooth function near 0. One of the main steps of the proof is to obtain a CR distribution u defined on a CR submanifold $\tilde{M}$, which is the parametrization of a product space $\omega_0 \times V$, such that u cannot be represented by a continuous function near the origin, where ω_0 is an open neighborhood of 0 in Ω. Furthermore, this is done such that any CR distribution on $\tilde{M}$ which does not depend nontrivially on to the variables of V, will (as a consequence of Theorem 3.0.11 mentioned) be a solution with respect to L.

If (Ω, L) is hypocomplex at 0 then the following property holds true:

(A) *For any given solution u (with respect to L) near 0 there exists a local hypoanalytic chart Z (taking values in $\mathbb{C}^n$ and such that L underlies Z) defined on an open neighborhood $\omega \subset \Omega$ of 0, and a holomorphic function $\tilde{u}$ defined on a neighborhood of $Z(0)$ such that $u = \tilde{u} \circ Z$ on ω.*

Since property (A) implies both hypocomplexity at 0 and the property that solutions near 0 which cannot be represented by a smooth function near 0, cannot be supported, the main result implies the following corollary.

Corollary 3.2.2. *Let (Ω, L) define a locally integrable structure, where $\Omega \ni 0$ is an open subset of a real Euclidean manifold. Then the following assertions are equiv-*

alent: (i) Property (A) *holds true for* (Ω, L). *(ii)* (Ω, L) *is hypocomplex at* 0. *(iii) There is no solution with respect to* L *which is singular at* 0.

Chapter 4

About paper III: On a mixed uniqueness condition for CR functions

The problem of unique continuation of CR functions is well-known and in Appendix A.5 we give a very short and incomplete survey of some results which are related to the question which we shall treat. One particular version of the problem is, to find subsets such that, if the functions vanish to infinite order on those subsets, then they automatically vanish on an ambient open neighborhood of a given reference point. For holomorphic functions, vanishing to infinite order at a single reference point, is sufficient (i.e., on a zero dimensional manifold). In the case of general CR manifolds, the single reference point must be replaced by a more complicated subset. Here we consider a problem originating from Nirenberg [55], we recall here the formulation which can be found in Fornaess & Sibony [28].

Question: Let $\Omega \subset \mathbb{C}^n, n \geq 2$, be a domain with smooth boundary. Let γ be a smooth curve in $\partial\Omega$, transverse to the complex tangent space $T_p(\partial\Omega) \cap J_p T_p(\partial\Omega)$, for every $p \in \gamma$ (here J is the complex structure map on $T\mathbb{C}^n$ defined by J_p on each $T_p\mathbb{C}^n$). Does it hold true that if $f \in C^\infty(\overline{\Omega})$, holomorphic on Ω and all derivatives (in all directions) $f^{(k)}$ vanish on $\gamma, \forall k \in \mathbb{Z}_+$, then $f \equiv 0$?

The version we consider is for $M \subset \mathbb{C}^n$, a C^∞-smooth generic CR manifold, but with additional geometric/growth conditions.

4.1 Some definitions

We recall the notion of transversality. We say that two manifolds intersect transversally, if at every point of intersection, their tangent spaces at that point generate the tangent space of the ambient manifold at that point, this also ensures that the intersection is a submanifold.

Definition 4.1.1 (See Baouendi & Zachmanoglou [12], p.9)**.** Let $\Omega \subset \mathbb{R}^N$ be an open set and let M and $\gamma \subset M$, be two differentiable submanifolds of Ω. We say that a continuous complex function f, defined on M, *vanishes to infinite order on* γ, if for every $\alpha \in \mathbb{R}$, the function,

$$z \mapsto f(z)(\text{dist}(z,\gamma))^{\alpha}, \tag{4.1}$$

is bounded in any compact set of M.

(For real-analytic functions vanishing to infinite order at a point implies vanishing on an open subset, but it is well-known that a smooth function can be nowhere real-analytic, see e.g. Kim & Kwon [45]). Here is an interesting example where unique continuation fails.

Example 4.1.2. Let

$$M := \{(z_1 = x_1 + iy_1, z_2) \in \mathbb{C}^2 : z_2 = te^{-2x_1}, t \in \mathbb{R}\}, \tag{4.2}$$

which induces the tangential CR-operator in the (z_1,t)-coordinates,

$$\overline{\partial} = \frac{\partial}{\partial \bar{z}_1} + t\frac{\partial}{\partial t}. \tag{4.3}$$

Define now,

$$\begin{cases} f \equiv 0 & t \leq 0, \\ f = \exp\left(-\frac{e^{2x_1}}{t}\right), & t > 0, \end{cases} \tag{4.4}$$

which is annihilated by $\frac{\partial}{\partial \bar{z}_1} + t\frac{\partial}{\partial t}$, thus a CR function on M. Then f vanishes to infinite order at 0, but not identically.

4.2 Result

Definition 4.2.1. Let M be a C^1-smooth N-dimensional real manifold. We define a set $\mathsf{C}(q) \subset M$ to be a *truncated double cone in M at $q \in M$* if there exists a parametrization of M by local Euclidean coordinates $(x_1,\dots,x_N)$ centered at q, such that $\mathsf{C}(q)$ is parametrized, in the variables $(x_1,\dots,x_N)$, by a truncated double cone at q in $\mathbb{R}^N$.

Proposition 4.2.2. *Let $M \subset \mathbb{C}^n$, be a C^∞-smooth real hypersurface. Let $\gamma \subset M$ be a real C^∞-smooth curve such that,*

$$T_z^c M + T_z \gamma = T_z M, \quad z \in \gamma. \tag{4.5}$$

Let $f \in CR^\infty(M)$ such that f vanishes to infinite order along γ. Assume that for each $q \in \gamma$, there exists a truncated double cone C at q in M, such that at least one of the following holds true:

(a) There is a constant $\theta \in \mathbb{R}$, such that $\mathsf{C} \subset \{|\mathrm{Re}(e^{i\theta} f)| \leq |\mathrm{Im}(e^{i\theta} f)|\}$.

(b) $\mathsf{C} \subset \{\mathrm{Re}\, f \geq 0\}$.

(c) $|f(z)|^{|z-q|} \to 0$, $z \to q$, $z \in \mathsf{C}$.

Then f vanishes on an M-neighborhood of γ.

In the proof we use the lower semi-continuity of maximal time parameters for the local flow of sections of $T^c M$ (see Theorem 1.0.2). Here is a short outline of the proof:

(1) Let $p_0 \in \gamma$. The transversality condition of our proposition, together with the aforementioned lower semi-continuity of maximal time parameters, implies that there is an open M-neighborhood of p_0, which can be covered by *Sussmann orbits*[1] of points of the intersection of γ with a ball centered at p_0.

(2) We prove that it is sufficient for a point $z \in \gamma$ to be minimal in order for $f \equiv 0$ on an M-open neighborhood of z.

(3) If $f \equiv 0$ on an M-open neighborhood of z then $f \equiv 0$ on an M-open neighborhood of the *Sussmann orbit*, $\mathcal{S}_z$, of z in M.

(4) If $z \in \gamma$ is non-minimal, we have two subcases: (a) every point of $\mathcal{S}_z$ is non-minimal. (b) $\mathcal{S}_z$ contains a minimal point.

(5) In the subcase (a), we first note that f vanishes on each member of the *local* CR orbit at z. Then we use the fact that there passes a complex hypersurface through every point of $\mathcal{S}_z$ thus vanishing of f will propagate to every point of $\mathcal{S}_z$ from a starting member of the local orbit at z.

[1] Here we mean the global Sussmann orbit with respect to the bundle, $T^c M$, whose sections are the CR vector fields. More generally one can speak of the Sussmann orbit with respect to any family L of vector fields, and in there exist cases where the global Sussmanns orbit can be open even if the local L-orbit is of lower dimension, see e.g. Baouendi et al. [4], p.72

(6) In subcase (b) we use a known result (due to Trepreau) which allows us to deduce that f has holomorphic extension to one side of M near z.

(7) We then prove that our given growth conditions forces f to vanish on $\mathcal{S}_z$.

(8) Thus in each of the subcases (a) and (b) we obtain that f vanishes on $\mathcal{S}_z$. This together with (1) will imply the wanted result as we let z vary in γ near the reference point p_0.

Chapter 5

About Paper IV: A local maximum principle for locally integrable structures

In paper IV we prove a local maximum principle for locally integrable structures. The proof uses the local version of the the result of Paper I (i.e. the maximum principle for continuous CR functions, see Theorem 2.2.1). The notion of (directional) Levi form has a generalization to locally integrable subbundles.

Definition 5.0.3. Let (Ω, L) be a locally integrable structure. Let $p \in \Omega$, let X, Y be two smooth sections of L over Ω. Let $T' \subset \mathbb{C} \otimes T^*\Omega$ be the subbundle satisfying $T' = L^\perp$ where $^\perp$ denotes the fiberwise defined orthogonal supplement with respect to the pairing between tangent vectors and cotangent vectors. By the *charactersitic set at* p, denoted T^0_p, we mean the fiber of $T^0 := T' \cap T^*\Omega$ at p. If Ω has dimension $n + l$, and l is the rank of L we set $r(z) := n - \dim(T^0_z)$. For a nonzero element $\xi \in T^0_p$, the map[1],

$$B^\xi_p(X, Y) := \langle \xi, \frac{1}{2i}[\widetilde{X}, \overline{\widetilde{Y}}]|_p \rangle, \tag{5.1}$$

(where $\widetilde{X}, \widetilde{Y}$ are any smooth sections of L, satisfying $\widetilde{X}_p = X, \widetilde{Y}_p = Y$) defines a form which is $\mathbb{C}$-linear in the first argument and anti-$\mathbb{C}$-linear in the second argument, on $L_p \times L_p$. The *Levi form at* p *in codirection* ξ for the locally

[1]The complex number on the right hand side is known to depend only on the values $\widetilde{X}_p = X, \widetilde{Y}_p = Y$, i.e., does not depend on choice of $\widetilde{X}, \widetilde{Y}$ satisfying this property, see e.g., Treves [72], p.46

integrable structure (Ω, L) is defined as the real valued Hermitian form,

$$\mathcal{L}_p^{\xi}(X) := B_p^{\xi}(X, X), \tag{5.2}$$

In the case of CR submanifolds of $\mathbb{C}^n$, recall the we used an embedability condition to define strictly pseudoconvex (see Definition 2.0.5), and it is a generalization of that notion which we use to define a convexity property for locally integrable structures, in Paper IV. First note that any generic m-dimensional CR submanifold $M \subset \mathbb{C}^n$ locally coincides with a hypoanalytic structure overlying $L = H^{0,1}M$, namely we can for each $p \in M$ find two open subsets $p \in U \subset \mathbb{C}^n$, $\omega^p \subset \mathbb{R}^m$ together with an embedding $Z : \omega^p \to \mathbb{C}^n$ such that (ω^p, L, Z) is a hypoanalytic chart with $U \cap M = Z(\omega^p)$.

Definition 5.0.4. Let $\mathfrak{E}$ denote the class consisting of all pairs (Ω, L), where $\Omega \subset \mathbb{R}^{n+\text{rank}L}$ is an open subset and $L \subset \mathbb{C} \otimes T\Omega$ a C^∞-smooth locally integrable subbundle satisfying $L \cap \overline{L} = 0$. It is known (see[2] Berhanu et al. [16], p.37) that in such case there is, for each point $q \in \Omega$, a local hypoanalytic structure (ω, L, Z), for a domain $q \in \omega \subset \Omega$, such that[3] $Z = (Z_1, \dots, Z_n) : \omega \to \mathbb{C}^n$ is an embedding onto a generic CR submanifold[4]. If there is such a choice of ω and Z such that Z embeds ω into a strictly pseudoconvex hypersurface in $\mathbb{C}^n$, then (Ω, L) will be called *strictly hypoanalytically pseudoconvex at* $q \in \Omega$.

Thus, for the subclass, $\mathfrak{E}$, of the class of locally integrable structures, Definition 2.0.5 has a straightforward analogue. In general, however, a local hypoanalytic chart induced by a locally integrable structure need *not* be an embedding. We shall in the more general case use the following (which can be shown to reduce to Definition 5.0.4 for the class $\mathfrak{E}$).

Definition 5.0.5 (Strict hypoanalytical pseudoconvexity, cf. Definition 2.0.5)**.** Let $\Omega \subset \mathbb{R}^{n+\text{rank}L}$ be an open subset and $L \subset \mathbb{C} \otimes T\Omega$ a C^∞-smooth locally integrable subbundle. (Ω, L), will be called *strictly hypoanalytically pseudoconvex at* $p \in \Omega$, if there exists a domain $\omega \subset \Omega$, $p \in \omega$ and a domain $V \subset \mathbb{R}^k$, $V \ni 0$, (for $k = 0$ we replace $\omega \times V$ with ω) together with an embedding $\widetilde{Z} : \omega \times V \hookrightarrow \mathbb{C}^{n+k}$, into a hypersurface, strictly pseudoconvex at $\widetilde{Z}(p, 0)$, such that:

[2] In more familiar terminology this result, except genericity, can be found in Baouendi et al. [10], p.37.

[3] Recall that by definition of a hypoanalytic chart, we must have $Z = (Z_1, \dots, Z_n)$ for $n = \dim\Omega - \text{rank}L$.

[4] It is furthermore known, see e.g. Boggess [21], p.103, that any generic CR submanifold of $\mathbb{C}^n$ can be locally parametrized by a sufficiently small choice of ω as the image of a hypoanalytic chart Z such that $Z(x, y) = x + i(y, \phi(x, y))$, $\phi(0) = d\phi(0) = 0$, where $x_1, \dots, x_n, y_1, \dots, y_{rankL}$, are *Euclidean* coordinates for ω.

(i) $\widetilde{Z} = (Z, Z')$, $Z' := (\widetilde{Z}_{n+1}, \ldots, \widetilde{Z}_{n+k})$, where (ω, L, Z) a hypoanalytic structure with global chart Z.

(ii) There is a C^∞-smooth coordinate system $x_1, \ldots, x_n, y'_1, \ldots, y'_{\mathrm{rank}L-k}, \tilde{y}_1, \ldots, \tilde{y}_k$, for ω, centered at some $p_0 \in \omega$, and a Euclidean coordinate system $t_1, \ldots, t_k$, for V, centered at $0 \in V$, with respect to which, $Z = x + i(y', \phi(x, \tilde{y}, y'))$, $\phi \in C^\infty(\omega, \mathbb{R}^{2n-\dim\Omega+k})$, $d_x\phi(0) = 0$, $Z' = t + i\tilde{y}$.

(iii) $(\widetilde{Z}, \omega \times V)$ is a global hypoanalytic chart in a hypoanalytic structure such that the image of $\widetilde{Z}$ is a generic submanifold.

It follows (see e.g. Baouendi et al. [10], p.10) from the properties of $\widetilde{Z}$ in Definition 5.0.5, that, that $\widetilde{Z}(\omega \times V)$ is a generic CR submanifold of $\mathbb{C}^{n+k}$, of CR dimension l. When $k = 0$ in Definition 5.0.5, we obtain the case of embeddings (when $k = 0$ we necessarily have $\widetilde{Z} = Z$ thus Z must itself be an embedding into $\mathbb{C}^n$), and the latter case certainly includes all locally integrable structures which underlie embedded CR submanifolds of $\mathbb{C}^n$. Let us look at an example where $k = 1$.

Example 5.0.6. Let $\omega = \mathbb{R}^4$ and let (x_1, x_2, y_1, y_2) denote Euclidean coordinates for ω. Obviously, $Z_1 = x_1 + iy_1$, $Z_2 = x_2 + i\phi(x_1, x_2, y_1, y_2)$, with $\phi = x_1^2 + y_1^2 + x_2^2 + y_2^2$, defines a global hypoanalytic chart on ω. We can define a global hypoanalytic chart $\widetilde{Z} : \omega \times \mathbb{R} \to \mathbb{C}^3$, introducing the Euclidean coordinate $t \in \mathbb{R} =: V$, centered at 0, by,

$$\begin{aligned} \widetilde{Z}_1 &= x_1 + iy_1, \\ \widetilde{Z}_2 &= x_2 + i(x_1^2 + y_1^2 + x_2^2 + y_2^2), \\ \widetilde{Z}_3 &= t + iy_2. \end{aligned} \tag{5.3}$$

In this example we can thus identify $k = 1$, $n = 2$, $\tilde{y} = y_2$, $y' = y_1$. To see that the image $\widetilde{Z}(\omega \times V)$ is a strictly pseudoconvex hypersurface in $\mathbb{C}^3$ at 0, we can introduce complex coordinates $(z_1, z_2, z_3) \in \mathbb{C}^3$ where $z_1 = x_1 + iy_1$, $z_3 = x_2 + i\,\mathrm{Im}\, z_3$, $z_2 = t + iy_2$, in order to obtain a defining function, $\rho(z_1, z_2, z_3) := (x_1^2 + y_1^2 + x_2^2 + y_2^2) - \mathrm{Im}\, z_3$, for $\widetilde{M} := \widetilde{Z}(\omega \times V)$. We have,

$$\left[\frac{\partial^2 \rho}{\partial z_k \partial \bar{z}_j}\right]_{1 \le k,j \le 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \tag{5.4}$$

thus $\sum_{k,j=1}^3 \frac{\partial^2 \rho(0)}{\partial z_k \partial \bar{z}_j} w_k \bar{w}_j > 0$, for all nonzero w such that $\sum_{j=1}^3 w_j \frac{\partial}{\partial z_j}|_0 \in H_0^{0,1}\widetilde{M}$ (and this is sufficient, see e.g. Boggess [21], p.164). Hence for $L = dZ^\perp$ we obtain that (ω, L) is strictly hyponanlytically pseudoconvex at 0.

Let us verify that there are locally integrable structures which do not underlie CR submanifolds of some $\mathbb{C}^n$ but which are nowhere hypoanalytically strictly pseudoconvex.

Example 5.0.7. Let $\Omega = \mathbb{R}^4$, and let (Ω, L) be the locally integrable structure such that L is spanned, when (x_1, x_2, y_1, y_2) are Euclidean coordinates for Ω, by ,

$$\begin{aligned} L_1 &= \frac{\partial}{\partial y_1} - i\frac{\partial}{\partial x_1}, \\ L_2 &= \frac{\partial}{\partial y_2}. \end{aligned} \tag{5.5}$$

It can be easily checked (see e.g. Berhanu et al. [16], p.24) that L underlies a hypoanalytic chart map $Z = (Z_1, Z_2) : \Omega \to \mathbb{C}^2$, defined by,

$$\begin{aligned} Z_1(x_1, x_2, y_1, y_2) &= x_1 + iy_1, \\ Z_2(x_1, x_2, y_1, y_2) &= x_2. \end{aligned} \tag{5.6}$$

We will look at this particular case of Z first. Note that there are only four possible decompositions $y = (y', \tilde{y})$, namely one can choose: $y' = y_1$, $y' = y_2$, $y = \tilde{y}$ or $y = y'$. Now assume there exists a hypoanalytic chart $\widetilde{Z} : \Omega \times V \to \mathbb{C}^{2+k}$, where $V \ni 0$ is a domain in $\mathbb{R}^k$, k a positive integer, such that $\widetilde{Z}(\Omega \times V)$ is a CR submanifold of $\mathbb{C}^{2+k}$, strictly pseudoconvex at $\widetilde{Z}((p, 0))$, for some point $p \in \Omega$, and such that $(\widetilde{Z}_3, \ldots, \widetilde{Z}_{2+k}) = t + i\tilde{y}$ where t is a real analytic coordinate system for V (note that V must be of dimension at most $k = 2$ because $y = (y_1, y_2)$). The case $k = 0$ is trivial since then it is impossible that strict hypoanalytical pseudoconvexity holds true here because (Ω, L, Z) does not define a generic CR manifold. So consider the case $k = 1$. We choose $y = (y', \tilde{y}) = (y_1, y_2)$ in Definition 5.0.5. This yields a graph representation of $\widetilde{Z}(\Omega \times V)$ over $\Omega \times V$, namely,

$$\begin{aligned} \widetilde{Z}_1(x, y, t) &= x_1 + iy_1, \\ \widetilde{Z}_2(x, y, t) &= x_2 \\ \widetilde{Z}_3(x, y, t) &= y_2 + it_1 \end{aligned} \tag{5.7}$$

This defines a generic CR submanifold which can be embedded in the flat hypersurface $\{\zeta \in \mathbb{C}^{n+k} : \operatorname{Im}\zeta_2 = 0\} \subset \mathbb{C}^{n+k}$. Whence $\widetilde{Z}(\Omega \times V)$ cannot be strictly pseudoconvex. Note that the only thing which is required for strict pseudoconvexity to fail above, is that, disregarding y_2, the hypersurface in $\mathbb{C}^2$ defined by,

$$\begin{aligned} Z_1(x_1, x_2, y_1) &= x_1 + iy_1, \\ Z_2(x_1, x_2, y_1) &= x_2. \end{aligned} \tag{5.8}$$

is not strictly pseudoconvex at any point. This also shows that no choice of $\widetilde{Z}$ with $k = 2$, can yield an image which is strictly pseudoconvex at any point.

Hence, there is for the given Z, no choice of $\tilde{Z}$ such that $\widetilde{Z}(\Omega \times V)$ is strictly pseudoconvex at any point.
For the general case of Z being a C^∞-smooth hypoanalytic chart overlying L, and simultaneously satisfying the given criteria of Definition 5.0.5, we see that each C^∞-smooth $\widetilde{Z}_j$, must be annihilated by $L_1 = -2i \cdot \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial y_1}\right)$, and is thus holomorphic with respect to $\eta := x_1 + iy_1 \in \mathbb{C}$. For fixed $(x_2, t, y_2) \in \mathbb{R}^3$, the restriction map $\widetilde{Z}|_{(x_1,t,y_2)} : \mathbb{C} \to \mathbb{C}^{n+k}$, defines a one dimensional complex manifold parametrized by $\eta = x_1 + iy_1 \in \mathbb{C}$. As the image $\widetilde{Z}(\Omega \times V)$ is foliated by complex manifolds it cannot be embedded in a hypersurface in $\mathbb{C}^3$ that is strictly pseudoconvex at any point[5]. Whence (Ω, L) is nowhere hypoanalytially strictly pseudoconvex. By Proposition 5.1.1, this implies that the local maximum principle holds true for any continuous solution f, with respect to L, defined on Ω. This completes Example 5.0.7.

5.1 Result

The main result of Paper IV is the following.

Proposition 5.1.1. *Let $\Omega \subset \mathbb{R}^N$ be an open subset, for a positive integer N, and let $L \subset \mathbb{C}\otimes T\Omega$ a C^∞-smooth locally integrable subbundle. If (Ω, L) is nowhere strictly hypoanalytically pseudoconvex, then for any sufficiently small domain $\omega \Subset \Omega$, and any $f \in C^1(\omega) \cap C^0(\overline{\omega})$ such that f is a solution with respect to L, it holds true that,*

$$\max_{z\in\partial\omega} |f(z)| = \max_{z\in\overline{\omega}} |f(z)|. \tag{5.9}$$

The proof of Proposition 5.1.1 is based upon contradiction and divided into three steps. We first locally embed the starting manifold into a generic CR submanifold $\widetilde{M}$ of possibly larger dimension. We then deduce (under the assumption that the statement of the main theorem fails) a counterexample to the local maximum principle for $\widetilde{M}$. The final step is to observe that $\widetilde{M}$ must satisfy the local maximum principle, because $\widetilde{M}$ is nowhere strictly pseudoconvex, whenever this is true for Ω in the sense of Definition 5.0.5.

5.1.1 A relation to Levi curvature

One possible way to define directional strict *Levi* pseudoconvexity for locally integrable structures (based upon a definition found in Tumanov [77], p.448,

[5]It is known that a strictly pseudoconvex hypersurface cannot contain a complex subvariety of any positive dimension, see e.g. D'Angelo [25], p.99.

there stated for generic CR submanifolds) is the following.

Definition 5.1.2. Let (Ω, L) be a locally integrable structure and let $p_0 \in \Omega$. Let $\xi \in T^0_{p_0} \setminus \{0\}$ be a characteristic codirection. (Ω, L) will be called *strictly Levi pseudoconvex, in codirection* ξ*, at* $p_0 \in \Omega$ if its Levi form with respect to the codirection ξ, at p_0, is definite (in the sense that the matrix associated to the Levi form in codirection ξ has only nonzero eigenvalues, which are all of the same sign).

Claim 5.1.3. *If* $(\Omega, L) \in \mathfrak{E}$*, and* $q \in \Omega$*, then the following assertions are equivalent:*

1. (Ω, L) *is strictly hypoanalytically pseudoconvex at* q *(in the sense of Definition 5.0.5).*
2. (Ω, L) *is strictly Levi pseudoconvex at* q *in some nonzero characteristic codirection at* q*.*
3. (Ω, L) *underlies, locally near* q*, a hypoanalytic structure,* (ω, L, Z)*,* $q \in \omega$*, such that* Z *is an embedding onto a generic* CR *submanifold, strictly pseudoconvex at* $Z(q)$*.*

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Appendix A

Appendix

A.1 Basic CR geometry

A.1.1 CR manifolds and CR functions

For prelimininaries on CR geometry see, e.g., Boggess [21], Zampieri [78], and for comprehensive reviews on the subject of holomorphic extension in CR geometry see Merker & Porten [52], Porten [57].

Let V be an even dimensional vector space over $\mathbb{R}$. We can turn it into a vector space over $\mathbb{C}$, if we define an $\mathbb{R}$-linear map $J\colon V \to V$ such that $J^2 = -Id$, such that,

$$(x + iy)v := xv + y(Jv), \quad v \in V, \quad x, y \in \mathbb{R}. \tag{A.1}$$

We denote the real tangent space of a real manifold M at $p \in M$ by T_pM. A real linear map J on an even dimensional real vector space V is called a complex structure if $J^2 = -Id$. A complex structure J on $T\mathbb{C}^n$ is defined as a real linear map $J\colon T\mathbb{C}^n \to T\mathbb{C}^n$ such that $J^2 = -Id$, specifically J is defined fiberwise on the tangent vector spaces by $\mathbb{R}$-linear maps $J_p\colon T_p\mathbb{C}^n \to T_p\mathbb{C}^n$. If $M \subset \mathbb{C}^n$ is a submanifold, $T_p^cM := T_pM \cap J_p(T_pM)$ is called the holomorphic tangent space of M at p. J maps each T_p^cM to itself thus defines a complex structure on T_p^cM. Recall that we can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, namely starting from $\mathbb{C}^n$ we can choose local coordinates x_j and y_j, $1 \leq j \leq n$, such that $z_j = x_j + iy_j$ are holomorphic coordinates (centered at p) for $\mathbb{C}^n$. Given the mappings,

$$(x, y) \mapsto (x + iy, x - iy) =: (z, \overline{z}), \quad (dx, dy) \mapsto (dx + idy, dx - idy), \tag{A.2}$$

(for $x, y \in \mathbb{R}^n$ and $z \in \mathbb{C}^n$) with inverse given by,

$$(z, \overline{z}) \mapsto \left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2} \right), \quad z \in \mathbb{C}^n, \tag{A.3}$$

the local coordinates yield a basis for the real tangent space at $p \in \mathbb{R}^{2n}$, according to $\frac{\partial}{\partial x_j}|_p, \frac{\partial}{\partial y_j}|_p$, $1 \leq j \leq n$. Let V be a real vector space with complex structure map J. J extends to be a $\mathbb{C}-$linear on the complexification (e.g., for $V = T_p\mathbb{C}^n$ we get the complexified tangent space $\mathbb{C} \otimes T_p\mathbb{C}^n$), namely any vector of the complexification belongs to the set $\{v + iw : v, w \in V\}$, so we define the extension according to,

$$v + iw \mapsto J(v) + iJ(w), \quad v, w \in V, \tag{A.4}$$

and the extension which we denote again by J satisfies $J^2 = -Id$. The extension will have the eigenvalues $\pm i$ because any eigenpair v, λ must satisfy $J^2 v = J(\lambda v) = \lambda^2 v = -v$. We denote the eigenspace corresponding to i by $V^{1,0}$ (in particular when $V = T_p\mathbb{C}^n$ we write $V^{1,0} = T_p^{1,0}\mathbb{C}^n$) and the eigenspace corresponding to $-i$ by $V^{0,1}$ (for $V = T_p\mathbb{C}^n$, this is $T_p^{0,1}\mathbb{C}^n$, and in that case $T_p^{0,1}\mathbb{C}^n$ is generated by $\{\frac{\partial}{\partial \bar{z}_j}|_p,\ 1 \leq j \leq n\}$). Explicitly we have a decomposition into $\mathbb{C}$-linear and anti-$\mathbb{C}$-linear parts,

$$v = \frac{v - iJ(v)}{2} + \frac{v + iJ(v)}{2}, \quad v \in V. \tag{A.5}$$

Note that $\overline{V^{1,0}} = V^{0,1}$. Let M be a real smooth submanifold of $\mathbb{C}^n$. $T_p^c M$ is a J_p invariant subspace of T_pM. If we consider the complex manifold $\mathbb{C}^n$ we have after extending J_p to the complexification,

$$\mathbb{C} \otimes T_p\mathbb{C}^n = T_p^{1,0}\mathbb{C}^n \oplus T_p^{0,1}\mathbb{C}^n, \tag{A.6}$$

This gives rise to the bundle, $T^{1,0}\mathbb{C}^n = \bigcup_{p \in M} T_p^{1,0}\mathbb{C}^n \subset \mathbb{C} \otimes T_p\mathbb{C}^n$. Performing the analogous splitting of $\mathbb{C} \otimes T_p^c M$ yields the decomposition into eigenspaces of J according to,

$$\mathbb{C} \otimes (T_pM \cap J_p(T_pM)) = H_p^{1,0}(M) \oplus H_p^{0,1}(M). \tag{A.7}$$

It is important that $H_p^{1,0}(M) \cap H_p^{0,1}(M) = \{0\}$. The sections of $H^{0,1}$ on an open subset, are the set of *tangential CR vector fields* on that subset. The maps J_p on $\mathbb{C} \times T_p\mathbb{R}^{2n}$ restrict to maps on $\mathbb{C} \otimes (T_pM \cap JT_pM)$ with $J_p^2 = -Id$.

Definition A.1.1. Let M be a real submanifold of $\mathbb{C}^n$.

- M is called a *CR manifold* if $\dim_{\mathbb{C}} (T_pM \cap J(T_pM))$ is constant for all $p \in M$.

- M is called *generic* if for every p in M,

$$(T_pM + JT_pM) = T_p\mathbb{C}^n, \tag{A.8}$$

(i.e., $A \in T_p\mathbb{C}^n \Rightarrow A = X + JY$, $X, Y \in T_pM$)

- The *real codimension* of a generic CR manifold M is the real dimension of $T_pM/(T_pM \cap J_pT_pM)$.

A useful fact in higher real codimension is that it is possible to locally describe generic CR manifolds as graphs over their tangent space. In particular we have the following theorem, see Boggess [21].

Theorem A.1.2 (Local graph representation theorem.)**.** *Let M is a smooth generic CR submanifold of $\mathbb{C}^n$ of real dimension $2n-d$, $d \leq n$, and let $p \in M$. Then there is an open neighborhood U of p, a nonsingular complex affine map $A\colon \mathbb{C}^n \to \mathbb{C}^n$ and a C^1-smooth function $h\colon \mathbb{R}^d \times \mathbb{C}^{n-d} \to \mathbb{R}^d$, s.t.,*

$$A(M \cap U) = \{(x+iy, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d} : y = h(x,w)\}, \tag{A.9}$$

where $h(0) = 0$, $\nabla h(0) = 0$.

Proof. Assume that $p = 0$, by translation. We show that there is a nonsingular complex linear map $A\colon \mathbb{C}^n \to \mathbb{C}^n$ that maps $T_0(M)$ to $\{\mathbb{R}^d \ni \operatorname{Im} z = 0\} \subset \mathbb{R}^{2n}$. If this holds then the graphing function for $A\{M\}$ satisfies the criteria for the function h. We know that $T_0M/(T_0M \cap JT_0M)$ is a real vector space dimension d and can be identified as a real vector subspace of $T_0\mathbb{C}^n$. We let $b_1, \ldots, b_d$ be an orthonormal basis for $T_0M/(T_0M \cap JT_0M)$ with respect to the induced inner product from $T_0\mathbb{C}^n$. Then the J-invariant space $(T_0M \cap JT_0M)$ is orthogonal to $\mathcal{A}_0 := T_0M/(T_0M \cap JT_0M)$ and $J\{\mathcal{A}_0\}$. Now J has only the imaginary eigenvalues $\pm i$ and $Ja \cdot b = -a \cdot Jb$, $a, b \in T_0(\mathbb{R}^{2n})$. If M is generic we have an orthonormal basis for $T_0(\mathbb{R}^{2n}) = T_0(M) \oplus J\mathcal{A}_0$, consisting of,

$$\{b_1, \ldots, b_d, Jb_1, \ldots, Jb_d\}. \tag{A.10}$$

We set $z = x + iy \in \mathbb{C}^d$, $w = u + iv \in \mathbb{C}^{n-d}$, and,

$$A(b_j) := \frac{\partial}{\partial x_j}, \quad A(Jb_j) := \frac{\partial}{\partial y_j}, \quad 1 \leq j \leq d, \tag{A.11}$$

$$A(b_j) := \frac{\partial}{\partial u_{j-d}}, \quad A(Jb_j) := \frac{\partial}{\partial v_{j-d}}, \quad d+1 \leq j \leq n. \tag{A.12}$$

Then A is complex linear, nonsingular and

$$A\{T_0M\} = \{(x, 0, u, v) : x \in \mathbb{R}^d, u, v \in \mathbb{R}^{n-d}\} = \{y = 0\}. \tag{A.13}$$

This completes the proof. □

We recall the following.

Definition A.1.3 (See e.g. Jost [44], p.41)**.** Let E and M be C^∞-smooth manifolds and let $\pi : E \to M$ be a C^∞-smooth map between the C^∞-smooth manifolds, such that each $E_x := \pi^{-1}(x)$, for $x \in M$, carries the structure of an r-dimensional real (complex) vector space.[1] Assume further that for each $x \in M$ there is an open $U \ni x$ and a diffeomorphism $\varphi : \pi^{-1}(U_x) \to U_x \times \mathbb{R}^r$ (for a complex vector bundle, $U_x \times \mathbb{C}^r$) such that for each $y \in U_x$ the map $\varphi_y := \varphi|_{E_y}$ is a vector space isomorphism (i.e. a bijective linear map) mapping $\pi^{-1}(y)$ to $\{y\} \times \mathbb{R}^r$ (for a complex vector bundle, $\{y\} \times \mathbb{C}^r$). The maps φ are called *local trivializations*. Then (E, π, M) is called a *vector bundle of rank r* and E is called the *total space*. A C^k-smooth *section*, s, of E over $\Omega \subset M$, is a C^k-smooth map $s : \Omega \to E$ such that $\pi \circ s = \mathrm{id}_\Omega$ (i.e $s(x) \in E_x$ for all $x \in \Omega$). The space of C^k-smooth sections of E over M is denoted $\Gamma^k(M, E)$.

Definition A.1.4 (CR function)**.** Given a CR submanifold $M \subset \mathbb{C}^n$, a function $f\colon M \to \mathbb{C}$ is called a *CR function (or distribution)* if it is annihilated (i.e., $Xf = 0$) by all tangential CR vector fields, i.e., sections X of $H^{0,1}M$. For distributions one defines f to be CR if $Xf = 0$, for every section X of $H^{0,1}M$ in the sense of distributions, i.e., $\langle f, X^{\mathrm{adj}}\phi\rangle$ for all test functions ϕ. A differentiable function f on a CR manifold of class C^1, is a CR function if $df|_{TM\cap JTM}$ is J-linear.

We can compare the definition of CR functions with that of holomorphic functions, namely $f \in C^1(\Omega), \Omega \subset \mathbb{C}^n$, a domain, is holomorphic if and only if $df = \partial f$, i.e., $\overline{\partial} f = 0$. In particular this means that df will be $\mathbb{C}$-linear. It is a generalization of this property which we have used in one of the definitions of C^1-smooth CR functions.

A smooth generic CR submanifold $M \subset \mathbb{C}^n$, $0 \in M$, has in local coordinates $(z = x + iy, w) \in \mathbb{C}^d \times \mathbb{C}^m$ a representation near 0, given by $M = \{y = h(x, w)\}$, $h(0) = \nabla h(0) = 0$. In general a basis for $H^{0,1}M$ near the origin is given by vector fields of the form (see e.g., Boggess [21], p.109, for the existence of such a representation),

$$X_j = \frac{\partial}{\partial \overline{w}_j} - i \sum_{k=1}^{d} \varphi_j \frac{\partial}{\partial \overline{z}_k}, \quad j = 1, \dots, m, \tag{A.14}$$

where φ_j is a specific function of the elements of the Jacobians of h with respect to x and w. All real-analytic CR functions on real-analytic CR submanifolds have local holomorphic extension (this is a result of Tomassini [75],

[1]Note that $E = \bigcup_{x\in M} E_x$, so in some sense M paramterizes E, and we require a priori that E itself be a C^∞-smooth manifold.

for textbook version, see Boggess [21], p.149). It is known (the result can be found in e.g., Baouendi et al. [4], p.95) that M is minimal at p in the sense of Tumanov if and only if the real dimension of the local CR orbit at p equals $\dim_{\mathbb{R}} M$, due to the fact that the local CR orbit is a CR submanifold and has the same CR dimension as M. With regards to minimality we cite an important theorem on *holomorphic* wedge extension theorem for minimal CR manifolds. The theorem is due to Tumanov, [76] (and a global version is independently due to Jöricke [43] and Merker [50] respectively).

Theorem A.1.5 (Holomorphic wedge extension from minimal CR manifolds)**.** *If $M \subset \mathbb{C}^n$, is a generic CR submanifold which is minimal at p, then there exists an ambient wedge, W at p such that any CR function f on M extends to some $\tilde{f} \in \mathscr{O}(W)$.*

On the other hand we have the following theorem (it can be found in the textbook of Baouendi, Ebenfeldt & Rothschild [4], p.238):

Theorem A.1.6. *If M is not locally minimal at $p \in M$ then for every $k \in \mathbb{Z}_{\geq 0}$, there exists a CR function of class C^k defined in a neighborhood of p which does not extend holomorphically to any local wedge of edge M at p.*

Here *locally minimal* means that the local CR orbit at p is open. The envelope of holomorphy of a wedge $\mathcal{W}$ in $\mathbb{C}^n$ is a Riemann domain which is a Stein manifold spread over $\mathbb{C}^n$. It is known that if a function f is holomorphic on a wedge W with generic edge, M, has polynomial growth up to M then the weak boundary values $f^* \in \mathcal{D}'(M)$ along M for $f \in \mathscr{O}(W)$ is a CR-distribution and furthermore f is continuous up to any open subset of M on which f^* is continuous (see Baouendi et al. [10], p.186).

A.1.2 Foliations

Definition A.1.7 (See, e.g., Rovenskii [63], p.1)**.** Let M be a real n-dimensional manifold. By a k-dimensional *foliation* of class C^r we mean a family $\mathcal{U} = \{M_\alpha\}_{\alpha \in \mathfrak{A}}$, (where $\mathfrak{A}$ is an index set) of connected subsets of M such that

1. $\bigcup_\alpha M_\alpha = M$,
2. $M_\alpha \cap M_\beta = \emptyset$ when $\alpha \neq \beta$,
3. $\forall p_0 \in M$, there exists a C^r chart (U_{p_0}, ϕ_{p_0}), $p_0 \in U_{p_0}$, such that if $U_{p_0} \cap M_\alpha \neq 0$ then the components of $\phi_{p_0}(U_{p_0} \cap M_\alpha)$ are the following parts of parallel affine subspaces, $A_c := \{(t_1, \ldots, t_n) \in \phi_{p_0}(U_{p_0}) :$

$t_{k+1} = c_{k+1}, \ldots, t_n = c_n\}$, $c_{k+1}, \ldots c_n \in \mathbb{R}$, constants. $d = (n-k)$ is sometimes called the *codimension* of the foliation and each (U_{p_0}, ϕ_{p_0}) is called a *foliated chart*.

Given two foliated charts[2] $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ of a given atlas, we may write, for each $w \in U_\alpha \cap U_\beta$,

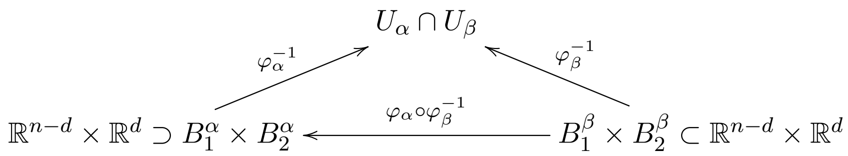
$$\varphi_\alpha(w) =: (x_\alpha(w), y_\alpha(w)) \in B_1^\alpha \times B_2^\alpha, \tag{A.15}$$

$$\varphi_\beta(w) =: (x_\beta(w), y_\beta(w)) \in B_1^\beta \times B_2^\beta, \tag{A.16}$$

where B_1^α, B_1^β are rectangular neighborhoods in $\mathbb{R}^{n-d}$ and B_2^α, B_2^β are rectangular neighborhoods in $\mathbb{R}^d$. For any choice of $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ it is possible to find (see Candel & Conlon [23], p.23) on $\varphi_\beta(U_\alpha \cap U_\beta)$ a C^∞-smooth coordinate change,

$$g_{\alpha\beta}(x_\beta, y_\beta) = \varphi_\alpha \circ \varphi_\beta^{-1}(x_\beta, y_\beta) = (x_\alpha(x_\beta, y_\beta), y_\alpha(y_\beta)), \tag{A.17}$$

i.e. $y_\alpha(x_\beta, y_\beta) = y_\alpha(y_\beta)$.



A less formal definition is the following.

Definition A.1.8 (see e.g. Anderson [2], p.6)**.** Let M be a real n dimensional manifold. A collection Σ of submanifolds of an open subset U of M is called a foliation of U by leaves of dimension m ($m < n$) if we can choose local coordinates $(t_1, \ldots, t_n)$ in U such that Σ is the collection of submanifolds $\Sigma_c = \{t_1 = c_1, \ldots, t_{n-m} = c_{n-m}\}$, $(c_1, \ldots, c_{n-m}) \in \mathbb{R}^{n-m}$. Let V be a neighborhood of $p \in \mathbb{C}^n$. A *complex foliation* Σ of $V \cap M$ is a foliation of $V \cap M$ such that each leaf of Σ is a complex submanifold of V.

Frobenius' theorem is a criterion for a given subbundle of the tangent bundle of an ambient space to itself define the tangent space of a real manifold. The theorem also has a complex version. A vector field $\sum_{k=1}^n a_k \frac{\partial}{\partial z_k}$ which is a section of $T^{1,0}\mathbb{C}^n$ is called a holomorphic vector field if each $a_j : \mathbb{C}^n \to \mathbb{C}$ is holomorphic. A holomorphic subbundle is a subbundle L of $T^{1,0}\mathbb{C}^n$ which is locally generated over $\mathscr{O}(\mathbb{C}^n)$ by m linearly independent holomorphic vector fields.

[2] In this text we follow the convention of e.g. Lang [47], p.20, that a chart map be bijective (as opposed to some texts which require only injectivity).

Theorem A.1.9 (Frobenius' theorem, complex version). *Let $L \subset T^{0,1}\mathbb{C}^n$ be an m-dimensional involutive ($[L, L] \subset L$) holomorphic subbundle and let $p_0 \in \mathbb{C}^n$. Then there is a neighborhood U of p_0 in $\mathbb{C}^n$ and a biholomorphic map $Z\colon U \to Z(U) \subset \mathbb{C}^n$ such that*

$$X(Z_j) = 0, \quad \forall X \in L, \quad 1 < j \leq n, \text{ on } U. \tag{A.18}$$

As pointed out in Boggess [21], p.58, this foliates U into complex manifolds each of dimension m. The following is due to Freeman [29] (for the concept of local straightening we refer the reader to Freeman [29] and Pinchuk [56]).

Theorem A.1.10. *If $M \subset \mathbb{C}^n$ is a generic Levi flat CR submanifold and $p \in M$ then there is a neighborhood V of p in $\mathbb{C}^n$ and a complex foliation Σ of $V \cap M$ with leaves of dimension $\dim_{\mathbb{C}} T^c_p M$, such that for any leaf Σ_c, and any point $q \in \Sigma_c$, $T^c_q M = T_q \Sigma_c$.*

A.1.3 The local approximation by entire functions

The following is a classic approximation theorem due to Baouendi & Treves [11], which provides local approximation of continuous CR functions, the original result concerns more general subbundles of vector fields but when applied in the case of the tangential Cauchy–Riemann vector fields one obtains the following theorem for continuous CR functions, see Boggess & Polking [20], p.761.

Theorem A.1.11 (Baouendi & Treves [11]). *Let $p_0 \in \Omega$, with $\Omega \subset \mathbb{C}^n$ a smooth generic submanifold. Then every neighborhood U of p_0 contains another neighborhood V of p_0 such that every continuous CR function on U is the uniform limit in V of holomorphic polynomials.*

The theorem has many generalizations, e.g., approximation by entire functions in L^p-sense of L^pCR functions, see Hounie & Malagutti [36], (for a textbook version, see Berhanu et al. [16], p.70).

Theorem A.1.12 (Generalized approximation in L^p sense). *Let $p_0 \in M$, with $M \subset \mathbb{C}^n$ a smooth CR submanifold. Then every neighborhood U of p_0 contains another neighborhood V of p_0 such that for every $L^p_{loc}CR$ function f on U, there is of a sequence of entire functions, $\{P_j\}_{j\in\mathbb{N}}$, which L^p converge to f on V.*

A.1.4 CR-hypoellipticity

In Theorem 3.0.3 the following definitions are used, and they can be found in the article by Porten & Nacinovich [53].

Definition A.1.13. Let M be an abstract CR manifold of CR dimension m and CR codimension d and let $U \subset M$ be an open subset. A smooth embedding $U \hookrightarrow \mathbb{C}^N, N = m + d$, given by $p \mapsto (z_1(p), \ldots, z_N(p))$ where the $z_1, \ldots, z_N$ are smooth CR functions, is called a *generic local CR-embedding*.

Definition A.1.14. Let M be an abstract CR manifold. M is called *CR-hypoelliptic at $p \in M$* if every CR-distribution near p is smooth on some neighborhood of p. M is said to have the *holomorphic extension property at $p \in M$* if there is a generic local CR-embedding $\phi : U \hookrightarrow \mathbb{C}^n$ ($U \subset M$ an open subset) such that for every CR distribution defined on a neighborhood $U' \subset U$ of p, there is a holomorphic function $\tilde{u}$ defined on a neighborhood V of $\phi(p)$, such that $\phi^* \tilde{u}$ is defined and equal to u on a neighborhood of p in U'.

A.2 Basic theory for hypoanalytic structures

For preliminaires on this subject see, e.g., Baouendi et al. [5] (where the concept is introduced) and also Treves [72], Berhanu [13], Baouendi & Treves [8], Treves [73]. We shall always assume that our manifolds are paracompact.

Definition A.2.1. Let Ω be a real smooth manifold. A *formally integrable structure L* on Ω is a complex vector subbundle of $\mathbb{C} \otimes T\Omega$ such that if $U \subset \Omega$ is open and if $X, Y \in \Gamma(U, L)$ are sections of L over U (i.e., $X_p, Y_p \in L_p$, for all $p \in U$) then so is $[X, Y]$. A formally integrable structure (Ω, L) such that $\forall p \in \Omega$, $L_p \cap \overline{L}_p = \{0\}$, is called a formal CR structure[3].

A complex tangent vector to M at p is a $\mathbb{C}$-linear map from the set of germs of smooth complex valued functions at p to $\mathbb{C}$, satisfying Leibniz rule.

Definition A.2.2 (See, e.g., Berhanu et al. [16], p.19)**.** Let Ω be a real N-dimensional manifold and let $L \subset \mathbb{C} \otimes T\Omega$ be a subbundle of rank l. Set $L_p^{\perp} := \{\xi \in \mathbb{C} \otimes T_p^*\Omega : \xi = 0 \text{ on } L_p\}$. L is called a *locally integrable structure* if for each $p_0 \in \Omega$ there is a neighborhood U of p_0 in M together with functions $Z_1, \ldots, Z_n \in C^{\infty}(U)$, $l = N - n$, such that $\text{span}\{dZ_{1p}, \ldots, dZ_{np}\} = L_p^{\perp}$, for all $p \in U$.

[3] A CR structure, L of rank l, on a manifold parametrized by an $(n + l)$-dimensional Ω, is locally integrable iff it can be embedded as a *generic CR* submanifold in $\mathbb{C}^n$, see Berhanu [16], p.37, or Baouendi & Rothschild [6], p.137.

Sometimes $L_p^\perp$ is denoted T'_p in the literature and this yields the bundle $T' := \bigcup_{p\in\Omega} T'_p$. Let Ω be a real smooth manifold of dimension $n+l$. A *hypoanalytic structure* on Ω is a collection $\mathcal{Z}$ of pairs (U_k, Z_k) where $U_k \subset \Omega$ is open and $Z_k := (Z_{k1},\dots,Z_{kn})\colon U_k \to \mathbb{C}^n$ a smooth map where $1 \leq n$ is independent of k, such that the following holds:

1. $\{U_k\}$ is an open cover of Ω.
2. $dZ_{k1},\dots,dZ_{kn}$ are $\mathbb{C}$-linearly independent at each point of U_k.
3. If $k \neq k'$ and $p \in U_k \cap U_{k'}$ there exists a holomorphic map $F^k_{k',p}$ of an open neighborhood of $Z_k(p)$ in $\mathbb{C}^n$ into $\mathbb{C}^n$ such that $Z_{k'} = F^k_{k',p} \circ Z_k$ in a neighborhood of p in $U_k \cap U_{k'}$.

The (U_k, Z_k) shall be called hypoanalytic charts. The span of the differentials define an orthogonal space, which will be a locally integrable subbundle $L \subset \mathbb{C} \otimes T\Omega$, with respect to the duality between complex tangent vectors and complex cotangent vectors, and we say that L underlies the hypoanalytic structure. Note that the span of the differentials cannot be of dimension higher than n, thus $l \leq n$. Given a real smooth manifold Ω it is clear that any locally integrable structure $L \subset \mathbb{C} \otimes T\Omega$ locally underlies at least one hypoanalytic structure. To each hypoanalytic structure for Ω one can associate a locally integrable structure L using that on U_k span$\{dZ^k_1,\dots,dZ^k_n\} = T'$ is subbundle of $\mathbb{C} \otimes T^*M$ (by which is meant that the fiber dimension of the subbundle equals n) together with the three properties in the definition of hypoanalytic structure. The orthogonal supplement, L, of the span of the differentials (with respect to the duality between tangent vector and cotangent vectors) is a locally integrable structure. For simplicity we shall denote a hypoanalytic structure by $(\Omega, L, \mathcal{Z})$ where by $\mathcal{Z}$ we denote the collection of hypoanalytic charts. A generic CR submanifold $\Omega \subset \mathbb{C}^n$ of CR dimension l defines a hypoanalytic structure where the real codimension is equal to $n-l$, and where $\mathcal{Z}$ consists of a single element. Given a local hypoanalytic chart (ω, Z), it is known, see Baouendi et al. [10], p.53, that for any choice of coordinates $u \in \mathbb{R}^{n+l}$ vanishing at q, such that the matrix $A(u) = [\partial Z_j(u)/\partial u_j]_{1\leq j\leq n}$ has nonzero determinant near q, one can introduce coordinates (s,t) such that $Z = s + i\psi(s,t)$, $Z(0) = q$, $\psi(0) = \partial_s\psi(0) = 0$, by setting $(s,t) = (s_1,\dots,s_n,t_1,\dots t_l)$ via, $s := \mathrm{Re}\, A(0)^{-1}Z(u)$, and $t_j := u_{j+n}$, $j = 1,\dots,l$. And this will satisfy $Z(0) = q$, $\phi(0) = \phi_x(0) = 0$. Two different hypoanalytic structures can overlie the same locally integrable structure, see Berhanu et al. [16], p.371, and Baounedi et al. [5], p.335, for examples. In general the mapping $Z = (Z_1,\dots,Z_n)\colon U \to \mathbb{C}^n$, (where $U \subset \Omega$ is an open subset) of a hypoanalytic chart need not be a diffeomorphism or even a local embedding. The differential df of a smooth function f is a section of $L^{\perp}$ if and only if $Xf = 0$, for each section X of L. The complex structure (in particular using the local holomorphic coordinates) of a complex analytic manifold is automatically a hypoanalytic structure. Let us relate this to CR submanifolds.

Example A.2.3. (See Baouendi et al. [5], p.335) Let Ω be a n-dimensional complex analytic manifold and assume $M \subset \Omega$ is a C^k-smooth N-dimensional submanifold in particular $N \leq 2n$. Assume the following holds: (*) *Given a complex analytic local chart* $(\tilde{U}, z)$ *in* Ω *the pull-backs*[4] *to* $U = \tilde{U} \cap M$ *of the differentials* $dz_1, \dots, dz_n$ *span an* m*-dimensional subspace of the complex cotangent space to* U *at every point of* U *(with* m *independent of the local chart).* Then M is a CR submanifold of Ω and has natural hypoanalytic charts $(U, Z_1, \dots, Z_m)$ with U as in (*) and the Z_j taken as the restriction to U, of m independent holomorphic functions in an open subset $\tilde{U} \subset \Omega$ such that $U = \tilde{U} \cap M$. M is called an embedded CR submanifold of Ω.

Definition A.2.4. Let Ω be a smooth real manifold, let $L \subset \mathbb{C} \otimes T\Omega$ be a locally integrable structure. Let $L_1, \dots, L_l$, define a local basis near a reference point p_0 for L. A distribution u defined on an open $p_0 \in V$ is called a solution on V if $L_j u = 0$ on V in the weak sense, $1 \leq j \leq l$. Let $Z \colon \omega \to \mathbb{C}^n$ be a local hypoanalytic chart overlying L on an open neighborhood ω of p_0 in Ω. A function $h \colon Z(\omega) \to \mathbb{C}$ is called a C^1-solution near p_0 if $h \circ Z$ is C^1 near p_0 and $L_j(h \circ Z) = 0, 1 \leq j \leq l$. [5]

It is common to introduce an appropriate basis for structure bundles as follows (see, e.g., Treves [72], p.11). Let Ω be a smooth real N-dimensional manifold and let (Ω, L) be a locally integrable structure and $0 \in \Omega$, where L has rank l. Letting $^{\perp}$ denote the orthogonal supplement, we have complex vector subspaces $L_p^{\perp} = \{\xi \in \mathbb{C} \otimes T_p^*\Omega : \xi = 0 \text{ on } L_p\}$ of each $\mathbb{C} \otimes T_p^*\Omega$, and a subbundle $T' \subset \mathbb{C} \otimes T^*\Omega$ defines as $T' := \bigcup_{p \in \Omega} L_p^{\perp}$. Each $V := L_p^{\perp}$ can be considered as a complex linear subspace of $\mathbb{C}^N (= \mathbb{C} \otimes T_p^*\Omega)$. We can decompose each n-dimensional complex V into $V = (V \cap \overline{V}) \oplus V_1$ for some linear subspace V_1 over $\mathbb{C}$. Set $V^0 := V \cap \mathbb{R}^N$. Since $V_1 \cap \overline{V}_1 = \{0\}$ the map Re: $z \mapsto \frac{z+\bar{z}}{2}$ maps $V \cap \overline{V}$ onto V^0 and its kernel is iV^0 thus, $V \cap \overline{V} = V^0 \oplus iV^0$, in particular $\dim_{\mathbb{R}} V^0 = \dim_{\mathbb{C}}(V \cap \overline{V}) =: d$. The map Re maps V_1 onto some $2r$-dimensional subspace of $\mathbb{R}^N$ which intersects V^0 only at $\{0\}$.

[4] Recall that if $\pi \colon A \to B$ is a differentiable map of manifolds and φ a differential 1-form on B then the pull-back of φ by π to A is $(\pi^*\varphi)(v) = \varphi(d\pi(v))$, $v \in T_pA$, so in the example, π is simply an embedding $\pi \colon U \to \tilde{U}$, and φ is some dz_j.

[5] The definition of a C^1-solution for a function on $Z(\omega)$ can be found for example in Marson [49]. Here we remind the reader that $Z(\omega)$ need not be a manifold.

Hence the map Re maps V onto a linear subspace of real dimension $2r + d$. If $\{e_1, \ldots, e_r\}$ are a linear basis of V_1 then $\{\bar{e}_1, \ldots, \bar{e}_r\}$ a linear basis of $\overline{V}_1$ and since $V_1 \cap \overline{V}_1 = \{0\}$ the vectors $e_j, \bar{e}_k$ are $\mathbb{C}$-linearly independent. Finally we can complement the basis for $V_1 + \overline{V}$ by adjoining a basis of real vectors $\{f_1, \ldots, f_d\}$ for V^0 such that the $e_1, \bar{e}_1, \ldots, e_r, \bar{e}_r, f_1, \ldots, f_k$ are all $\mathbb{C}$-linearly independent. Now at a given point $p \in \Omega$ the fiber $T'[p]$ has a linear basis of the kind $e_1^p, \bar{e}_1^p, \ldots, e_r^p, \bar{e}_r^p, f_1^p, \ldots, f_k^p$ with properties as above. This implies that in some neighborhood U of 0, T' is spanned by $2r + k$ smooth differential 1-forms $\varphi_1, \bar{\varphi}_1, \ldots, \varphi_r, \bar{\varphi}_r, \psi_1, \ldots, \psi_k$ where,

$$\varphi_1 \wedge \bar{\varphi}_1 \wedge \cdots \wedge \varphi_r \wedge \bar{\varphi}_r \wedge \psi_1 \wedge \cdots \wedge \psi_k \neq 0, \tag{A.19}$$

and where $\psi_1, \ldots, \psi_k$ are real at the point 0. Now for a sufficiently small neighborhood of 0 in U there are n smooth functions $Z_1, \ldots Z_n$ whose differentials also make up a basis of T'. We can assume all Z_j vanish at the origin. There exists a nonsingular complex matrix $[a_{jk}]_{1 \leq j,k \leq n}$ such that, at the origin, $\varphi_j = \sum_{i=1}^n a_{ji} dZ_i$, $\psi_k = \sum_{i=1}^n a_{(r+k)i} dZ_i$, for $1 \leq j \leq r$ and $1 \leq k \leq (n - r)$. Since the entires a_{ij} are constant we may substitute $(\sum_{i=1}^n a_{ji} Z_i)$ for Z_j, $1 \leq j \leq n$. In other words we may assume that $\varphi_j = dZ_j$, $\psi_j = dZ_{r+k}$ at the origin ($1 \leq j \leq r$ and $1 \leq k \leq (n-r)$). This implies that in a neighborhood of 0,

$$Z_1 \wedge \bar{Z}_1 \wedge \cdots \wedge Z_r \wedge \bar{Z}_r \wedge Z_{r+1} \wedge \cdots \wedge Z_n \neq 0, \tag{A.20}$$

and that $dZ_{r+1}, \ldots, dZ_n$ are real at the origin. Finally this means that we may regard,

$$x_j = \operatorname{Re} Z_j, \quad y_j = \operatorname{Im} Z_j, \quad 1 \leq j \leq r \tag{A.21}$$

$$s_k = \operatorname{Re} Z_{r+k}, \quad 1 \leq k \leq (n - r), \tag{A.22}$$

as part of a system of local coordinates near 0 centered at 0 (and possibly after having contracted U about 0 we assume the coordinates hold on U). We denote by $t_1, \ldots, t_{l-r}$ the remaining coordinates in that system. Recall that we substituted for Z_j, a linear combination as above, there exists a smooth real-valued functions $\phi_k(x, y, s, t)$ such that $Z_{r+k} = s_k + i\phi_k(x, y, s, t)$, $1 \leq k \leq (n - r)$. Since the $dZ_{r+1}, \ldots, dZ_n$ are real at the origin $\varphi_k(x, y, s, t)(0) = 0$, $d\varphi_k(x, y, s, t) = 0$, where $1 \leq k \leq n - r$, see for example Treves [72], p.39.

Repeating the first part of the analysis but with T' replaced by L we can obtain that there is a neighborhood of 0 in Ω together with a basis $L_1, \ldots, L_r, \overline{L}_1, \ldots, \overline{L}_r$, $L_{r+1}, \ldots, L_{r+l}$, for the sections of L on that neighborhood where l is the complex dimension of L and where $L_{r+1}, \ldots, L_{r+l}$ are real at 0. It is common to define $T^0 := T' \cap T^*\Omega$ (which in general need not be a vector bundle since the fiber dimension may vary).

A.3 The method of proof of Theorem 3.0.10

We briefly sketch the method of proof of Marson [49] for Theorem 3.0.10. The first step of the proof is to handle the case when 0 is nonminimal. In that case Marson uses the existence of a distribution solution u' near 0 (the existence of such solutions is shown in, e.g., Treves [72], p.93) such that for every open neighborhood U' of 0 there exists an open $U'' \subset U'$ on which u' vanishes identically. Marson [49] then proves as a lemma such a property is impossible for distribution boundary values of functions holomorphic on a special kind of wedge in the sense defined in Marson [49], and that u' induces such a distribution. This provides the necessity of minimality at 0 for wedge extension at 0.

So assume 0 is a minimal point. Let $\Omega_0 \subset \Omega$ be an open neighborhood of 0 and let (Ω_0, L, Z) and let (Ω_0, L, Z) define a hypoanalytic structure (with a single chart Z). Then there are local real analytic coordinates,
$(x_1, \ldots, x_r, s_1, \ldots, s_{n-r}, y_1, \ldots, y_l)$ for Ω_0 centered at the origin such that,

$$Z_j(x, s, y) = x_j + iy_j, \qquad 1 \leq j \leq r, \tag{A.23}$$

$$Z_{r+j}(x, s, y) = s_j + i\phi_j(x, s, y), \qquad 1 \leq j \leq n - r, \tag{A.24}$$

where ϕ is a smooth map with $\phi(0) = d\phi(0) = 0$ (see Appendix A.2). The method is to associate wedge extendability of (Ω_0, L, Z) to that of a generic CR manifold defined as follows. Let $0 \in A \subset \mathbb{R}^{l-r}$ be an open subset and set $\widetilde{\Omega}_0 = A \times \Omega_0$. Let $\widetilde{M}$ be the manifold parametrized by the map $\widetilde{Z} \colon \widetilde{\Omega}_0 \to \mathbb{C}^{n+l-r}$ according to,

$$\tilde{Z}_{r+j}(\tilde{x}, x, s, y) = Z_j(x, s, y), \qquad 1 \leq j \leq n, \tag{A.25}$$

$$\tilde{Z}_j(\tilde{x}, x, s, y) = \tilde{x}_{r+j} + iy_{r+j}, \qquad 1 \leq j \leq l - r, \tag{A.26}$$

$$\tag{A.27}$$

where $\tilde{x} \in A$ (i.e., is a real $(l-r)$-tuple) but for simplicity, the indices of $\tilde{x}$ begin with $r+1$. Then $\widetilde{M} \subset \mathbb{C}^{n+l-r}$ is a generic CR submanifold and a basis for the CR vector bundle $\widetilde{L} \subset \mathbb{C} \otimes T\widetilde{\Omega}_0$ is given by,

$$\tilde{L}_{r+j} = \frac{1}{2}\frac{\partial}{\partial \tilde{x}_{r+j}} + L_{r+j}, \qquad 1 \leq j \leq l - r, \tag{A.28}$$

$$\tilde{L}_j = L_j, \qquad 1 \leq j \leq r. \tag{A.29}$$

Marson [49] then proves that (Ω_0, L) is minimal at 0 if and only if $(\widetilde{\Omega}_0, \widetilde{L})$ is minimal at 0. The final step is to show that wedge extendibility at 0 for $(\widetilde{\Omega}_0, \widetilde{L})$ implies wedge extendibility of (Ω_0, L). This completes the sketch of the proof of Theorem 3.0.10.

A.4 Short survey of previous related results on the maximum principle for CR functions

We now survey some versions of the maximum modulus principle for CR functions on certain CR manifolds.

Definition A.4.1 (See Hill & Nacinovich [34], p.152)**.** An *abstract smooth almost CR manifold of type (n, k)* is a connected smooth paracompact manifold M of dimension $2n+k$, and a smooth subbundle B of TM of rank $2n$ and a complex structure J on the fibers of B. Defining by $T^{0,1}M$ the complex subbundle of the complexification of B given by $T^{0,1}M = \{X + iJX | X \in B\}$, we call M a CR manifold if $[\mathcal{C}^\infty(M, T^{0,1}M), \mathcal{C}^\infty(M, T^{0,1}M)] \subset \mathcal{C}^\infty(M, T^{0,1}M)$.

We shall first mention some results from Ellis et al. [26] and Iordan [41], and the related Hill & Nacinovich [35], where these results are generalized to abstract CR manifolds.

Definition A.4.2. $M \subset \mathbb{C}^n$ be a complex submanifold. A point $p \in M$ is called an *extreme point* of M if there is, in a neighborhood U of p, a local coordinate system $z = (z_1, \ldots, z_n)$ such that M is locally near p not contained in any $\mathbb{C}^k$, $k < n$, and $z(p) = 0$, $M \cap U \subset \{z | \operatorname{Im} z_n \geq 0\}$.

The absence of extreme points (see Definition A.4.2) is necessary in order for the strong maximum principle to hold. From Ellis et al. [26], p.711, we have the following result.

Theorem A.4.3. *If $p \in M$ is an extreme point of M there exists a normal direction $\xi \in N_pM$ such that the directional Levi form at p with respect to the direction ξ is positive semi-definite. On the other hand if for a given point $p \in M$ it holds true that there exists a normal direction ξ such that the directional Levi form at p with respect to that direction is strictly positive definite, then p is an extreme point of M.*

Theorem A.4.4 (Ellis et al. [26], p.712)**.** *The absence of extreme points is necessary for the strong maximum principle to hold (where by strong maximum principle it is meant that given an open connected $U \subset M$, a CR function on U cannot have a weak local maximum unless it is constant on u).*

In the case of a smooth hypersurfaces $M \subset \mathbb{C}^n$ when any CR function has local holomorphic extension (e.g., if at each point $p \in M$ the Levi form at p has two eigenvalues of opposite signs) then it is feasible to obtain that smooth CR functions on M satisfy the strong maximum modulus principle (i.e., a local maximum on a connected open subset implies a constant CR function), using that the modulus of the local extension is bounded by the maximum modulus of the CR function. It is known that a Levi flat hypersurface satisfies the weak (local) maximum principle but in general not the strong maximum principle. In Iordan [41] it is shown that every point of a totally real submanifold $M \subset \mathbb{C}^n$ is an extreme point of M. Also the following theorem is given.

Theorem A.4.5 (Iordan [41])**.** *If M is a CR submanifold of $\mathbb{C}^n$ without extreme points, then for any (differentiable) CR function f on M, $|f|$ cannot have a strong local maximum at any point of M.*

We also have the following result due to Berhanu & Wang [14].

Theorem A.4.6 (Berhanu & Wang [14])**.** *If M is a smooth generic CR submanifold of $\mathbb{C}^n$, and $f \in CR^2$ such that $|f(p_0)|$ is a strict local maximum for some $p_0 \in M$, then p_0 belongs to the closure of the set of strictly definite points (i.e., points where the $\mathcal{L}_{M,p_0}(X_{p_0}) \neq 0$, for all nonzero $X_{p_0} \in H^{1,0}_{p_0}M$).*

Definition A.4.7. A real valued C^2-smooth function g on M is called *L convex* (i.e., convex with respect to a vector bundle L) if the Levi form of g is strictly positive on every section of L.

If (M, L) admits an L convex function and if $\Omega \Subset M$ is open, then any $f \in C^2(\Omega) \cap C(\overline{\Omega})$ which is also a CR function on Ω, satisfies,

$$|f(z)| \leq \max_{w \in \partial\Omega \cup (\overline{A} \cap \Omega)} |f(w)|, \forall z \in \Omega, \tag{A.30}$$

where A is the set of points in M where the Levi form is strictly definite. Furthermore if h is a C^2-smooth CR function such that $|h(p)| > |h(q)|$ for all $q \neq p$, for some $p \in M$, then $p \in \overline{A}$. In Hill & Nacinovich [35] we have a maximum principle for smooth abstract weakly pseudoconcave CR manifolds if minimality holds locally where the principle is to be applied.

Definition A.4.8 (Hill & Nacinovich [35], p.105)**.** Let M be an abstract CR manifold and let $H^{1,1} \subset HM \otimes HM$ be the smooth subbundle whose fiber at each $z \in M$ is generated by elements of the form $v \otimes v + (Jv) \otimes (Jv)$, $v \in H_z M$. The Levi for is regarded as the Hermitian form associated to an appropriate choice of Hermitian metric on the fibers of the holomorphic tangent bundle, for the embedded smooth CR manifold case we always assume the Hermitian metric induced from the ambient $\mathbb{C}^n$. The Levi form induces a linear form $H^{1,1}M \to \mathbb{R}$ according to $\mathcal{L}^\xi(v \otimes v + (Jv) \otimes (Jv)) = \mathcal{L}^\xi(v)$. M is called *weakly pseudoconcave* if: $\forall p \in M$, there is an open neighborhood $U \subset M$, of p and a smooth section X of $H^{1,1}U$ such that, $\mathcal{L}_\xi(X) = 0$, $\forall z \in U$, and nonzero ξ belonging to the annihilator of $T^c_z M$ in in $T^*_z M$.

Note that their definition of weakly pseudoconcave does not coincide with the definition that the directional Levi form have at least one non-positive eigenvalue. In fact this definition for smooth CR submanifolds of $\mathbb{C}^n$, implies that the trace of the directional Levi form is 0, see Proposition 1.1, Hill & Nacinovich [35]. Denote by $\mathcal{F}(x_0, \Omega)$ the Sussmann leaf in Ω through x_0 where $(x_0 \in)\Omega \subset M$ is an open subset of an abstract CR manifold.

Theorem A.4.9. *Let M be a smooth abstract weakly pseudoconcave (in the sense of Hill & Nacinovich [35]) almost CR manifold of type (n, k). Let $\Omega \subset M$ be a nonempty open subset and $x_0 \in \Omega$. Let $u \in C^2(\mathcal{F}(x_0, \Omega))$ be a CR function on the almost CR manifold $\mathcal{F}(x_0, \Omega)$. If M is minimal at x_0 and $|u|$ has a local weak ($\geq$) maximum at x_0. Then u is constant along $\mathcal{F}(x_0, \Omega) \cap \Omega$.*

Furthermore we have in Hill & Nacinovich [35], Corollary 4.3, p.109, a theorem on only continuous CR functions, which is a form of identity theorem on Sussmann leaves.

Theorem A.4.10 (Hill & Nacinovich [35], p.110)**.** *Let M be a weakly pseudoconcave (in the sense of [35]) abstract almost CR manifold of type (n, k), and u a continuous CR function on M. Let $\Omega \subset M$ be a nonempty open subset and $x_0 \in \Omega$. If $u \equiv 0$ on $\mathcal{F}(x_0, M) \cap \Omega$ then $u \equiv 0$ along $\mathcal{F}(x_0, M)$.*

A consequence is a maximum principle which in the original article by Hill & Nacinovich, covers a more general class of CR manifolds and involves (imprecisely) that a function which is CR and C^2-smooth on the local orbit of a point p_0, cannot attain maximum at p_0 unless it is constant locally near p_0, on the closure of the orbit.

Theorem A.4.11 (Hill & Nacinovich [35], p.110)**.** *Let M be a smooth weakly pseudoconcave CR submanifold of $\mathbb{C}^n$. Let further $\Omega \Subset M$ be an open and proper subset and $u \in C^2(\mathcal{F}(p_0, \Omega))$ be a CR function (here one uses that the orbit is an immersed, but not necessarily embedded, CR submanifold). If $|u(p_0)| = \sup_{z \in \mathcal{F}(p_0, M) \cap \Omega} |u(z)|$ the u is constant along $\overline{\mathcal{F}(p_0, \Omega)} \cap \Omega$.*

Let $Q \subset M$ be an open subset and $R \subset \mathbb{C}^n$ a connected open subset such that $Q \subset \overline{R}$. We denote the set of C^∞-smooth CR functions on Q by $CR^\infty(Q)$ and we denote $CR^\infty(R \cup Q) := \{f \in C^\infty(R) : f \in \mathscr{O}(R \cup Q)\}$. Consider the restriction map $\varphi\colon CR^\infty(R \cup Q) \to CR^\infty(Q)$. The following is a proposition from Carlson & Hill [22], p.93.

Proposition A.4.12. *Let $f \in CR^\infty(Q)$. If φ is surjective, then for any function, $\tilde{F} \in CR^\infty(R \cup Q)$, satisfying $\tilde{F}|_Q = f$, $\sup_{z \in R \cup Q} \left|\tilde{F}\right| = \sup_{z \in Q} |f|$.*

Sibony [69], p.20, proves a maximum principle for C^3-smooth CR functions in the sense that given a domain $\omega \Subset M$, $M \subset \mathbb{C}^n$, a CR submanifold with defining functions $\rho_1, \ldots, \rho_d$, the CR^3 functions attain maximum on $\partial\omega$ or the set of points, $p \in M$, which satisfy that there exists j, such that, $\sum_{k,l=1}^n \frac{\partial^2 \rho_j}{\partial z_k \partial \bar{z}_l} w_k \bar{w}_l$, for each nonzero $w \in \mathbb{C}^n$ such that $\sum_j w_j \partial z_j \in H_p^{0,1} M$. When it comes to the *strong* maximum principle, one can recall how the proof for holomorphic functions, can be given such that it vitally depends upon the open mapping theorem. For the real analytic case we have the following.

Theorem A.4.13 (See Berhanu [15])**.** *Let $M \subset \mathbb{C}^n$, be a real analytic CR hypersurface which satisfies the strong maximum principle. Let further $p_0 \in M$ and f be a nonconstant holomorphic function near p_0. Then f is an open map near p_0.*

In a classical article Rossi [61] (see also Rossi [62]) Rossi gives a result on peak sets. A compact $K \subset X, X$-compact Hausdorff, is a peak set if there exists a complex valued function f such that $|f| = 1$, on K, but $|f| < 1$ on $X \setminus K$, and local peak set means that there is a neighborhood, U, of K such that f peaks relative to U (instead of X). Let $\mathscr{A} \subset C(X), X$-compact Hausdorff, be a subalgebra, let $M_{\mathscr{A}}$, the set of maximal ideals of $\mathscr{A}$, and let $S_{\mathscr{A}}$ be the Šilov boundary of $\mathscr{A}$ (this is the smallest closed subset of $S_{\mathscr{A}}$, on which every element of $\mathscr{A}$ attains its maximum). Rossi's results yield in particular the following maximum principle involving the polynomial convex hull (this particular formulation is found in Stout [70], p.78):

Theorem A.4.14. *If X is a compact in $\mathbb{C}^n$, if E is a compact subset of $\hat{X}$, U is an open subset of $\mathbb{C}^n$ containing E, and if $f \in \mathscr{O}$, then $\|f\|_E = \|f\|_{(E\cap X)\cup\partial E}$.*

We mention that Hill & Nacinovich [34], use a notion called essential pseudoconcavity and show that it is a useful condition with respect to the maximum principle.

Theorem A.4.15 (Hill & Nacinovich [34], p.276)**.** *Let M be a connected paracompact smooth essentially pseudoconcave abstract almost CR manifold of type (n, k), of finite type. Let further $\Omega \Subset M$ be an open and proper subset and $u \in C^0(\overline{\Omega}) \cap CR(\Omega)$. Then $|u(x)| \leq \max_{w\in\partial\Omega} |u(w)|$, for every $x \in \Omega$.*

For the strict definition of finite kind and essentially pseudoconcave we refer to Hill & Nacinovich [34]. As a corollary we have that if M is also compact then any global CR function is constant.

Theorem A.4.16 (Hill & Nacinovich [35], p.111)**.** *Let M be a smooth connected essentially pseudoconcave abstract-CR manifold of type (n, k). Let $u \in CR^2(\Omega)$. If $|u(x)|$ obtains weak local max at any point of M then u is constant on M.*

A.5 Short survey of previous related results on unique continuation of CR functions

One of the earliest uniqueness results is due to Luzin & Privalov [48] and as a special case implies that a a function $f \in \mathscr{O}(\Delta) \cap C(\overline{\Delta})$, which vanishes on a relatively open arc of $\partial\Delta$ automatically vanishes on Δ. Baouendi & Treves [8], propose so called noncharacteristic submanifolds as candidates subsets on which vanishing of a CR functions implies local unique continuation

Definition A.5.1. Let M be a real manifold and let T' be a complex subbundle of $\mathbb{C} \otimes T^*M$, such that T' is locally generated by exact differentials. Set $T^0 = T' \cap T^*M$ (the *characteristic* set). A *noncharacteristic* set submanifold, is a subset $\Sigma \subset M$ such that the conormal bundle, $N^*\Sigma$ does not intersect T^0 off the zero section.

Theorem A.5.2 (Vanishing on a generic submanifold, see Airapetyan [1], p.105)**.** *Let $M \subset \mathbb{C}^n$ be a generic CR submanifold and $N \subset M$ a generic submanifold. Then any continuous CR function on M such that $f|_N \equiv 0$, vanishes identically in some neighborhood of N.*

In Hunt [38] it is shown by way of examples that a local version of this theorem is also in some sense optimal.

Definition A.5.3. Let $N \subset \mathbb{C}^n$ be a real k dimensional submanifold and $p_0 \in M$. N is called *locally CR* at p_0 if $\dim_{\mathbb{C}} T^c_{p_0} N$ is constant near p_0 in N, and N is called *locally generic* at p_0 $\dim_{\mathbb{C}} T^c_{p_0} N = \max\{k-n, 0\}, \forall z$, near p_0.

Theorem A.5.4 (Continuation from locally generic submanifolds, see Hunt [38])**.** *Let $V \subset M$, be open, where $M \subset \mathbb{C}^n$ is a C^∞ CR submanifold. Assume that $N \subset V$ is locally generic at $p_0 \in N$, and a real manifold of dimension $k \geq n$. Then $\exists \omega$, an M-neighborhood of p_0 such that any continuous CR function, f, on M satisfying $f \equiv 0$, on N also must satisfy $f \equiv 0$, on ω.*

Counterexamples are given for the cases $k < n$ or non locally generic N.

Theorem A.5.5 (Schmalz [66])**.** *Let $M \subset \mathbb{C}^n$ be a generic CR submanifold of class C^3 and $p_0 \in M$ such that Tumanov's minimality condition holds at p_0. Let $N \subset M$ be a generic C^3 submanifold and $K \subseteq N$ a set such that the intersection of K with some given small neighborhood U of p_0 has positive Lebesgue measure on N. Then any continuous CR function vanishing on K vanishes identically on $M \cap U$.*

The proof is a consequence of the following theorem.

Theorem A.5.6. *Let W be a wedge with edge $M \subset \mathbb{C}^n$, M-a generic CR submanifold of class C^3. Let $N \subset M$ be a generic C^3 submanifold and $K \subseteq N$ a set of positive Lebesgue measure (on N). Then any continuous CR function vanishing on K vanishes identically.*

Idea of the proof. Let $\dim(N) = k$. It is possible to construct a smooth family of analytic discs such that: there exists a hypersurface $H \subset \mathbb{C}^k$, and a subset $H' \subset H$, of positive $(2n-1)$-Lebesgue measure, satisfying that $\forall p \in H'$, there is a disc passing p and some $l \subseteq \partial D, l$ of positive length such that the image of l belongs to K, see Sadullaev [64]. Schmalz then shows that the family of discs can be chosen such that their images lie in a wedge W. Then by the one-dimensional uniqueness theorem the theorem is proved.
Now applying Tumanov's wedge extension theorem (i.e., the possibility to obtain holomorphic extension to a wedge via analytic discs, under the condition of minimality), we have in Theorem A.5.5, a wedge due to minimality, which combined with the last theorem gives the vanishing result.

Theorem A.5.7 (See Schmalz [66], Theorem 5)**.** *Let $W \subset \mathbb{C}^n$, be a wedge with a CR submanifold $M(\ni 0) \subset \mathbb{C}^m \times \mathbb{R}^d$, as edge, such that M near 0 is given by $\{y = g(x,w)\}$, $g \in C^{3,\alpha}, \alpha > 0$(by this notation is meant that the third order derivatives of the defining function are α-Hölder continuous), $g(0,0) = 0 = \nabla g(0,0)$, and let $N = \{y = h(x)\}$, $h \in C^{3,\alpha}, h(0,0) = 0 = \nabla h(0,0)$, be a generic submanifold of M. Let $f \in \mathscr{O}(W) \cap C^0(M)$, and let m be a nonnegative, non-decreasing function on $[0,\infty)$ such that, $|f(z)| \leq m(|z|), \quad \forall z \in N$, $\int_0^\epsilon \log m(s)ds = -\infty$. Then $f \equiv 0$.*

Rosay [60] describes the relation of our considerations, in codimension one, to the Cauchy problem, i.e., do functions, u, annihilated by a complex vector field, $Lu \equiv 0$ together with the Cauchy data $u = 0$ on some characteristic hypersurface vanish identically in an ambient local neighborhood? It is shown that there exists a strictly pseudoconvex CR structure on $\mathbb{R}^3$, where it is possible to find, see Rosay [60], a smooth CR function $\equiv 0$, on an open subset, but not vanishing identically. We can mention the following result: Let $A(x,y)$ be a smooth function near $(0,0) \in \mathbb{R}^2$, let

$$\begin{cases} \frac{\partial u}{\partial x} + A(x,y)\frac{\partial u}{\partial y} \equiv 0, \\ u(0,y) \equiv 0, \end{cases} \tag{A.31}$$

Then $u \equiv 0$ if: A is real analytic, or; A is real-valued, or; $\mathrm{Im}A(0,0) \neq 0$. The above example is inspired by a counterexample of Cohen (for a despription of Cohen's example, see e.g. Rosay [60], p.294), who proves the existence of a smooth C^∞ function f near $0 \in \mathbb{R}^2$, such that, $\frac{\partial f}{\partial x} + A(x,y)f = 0$, for some smooth A near $0 \in \mathbb{R}^2$, and $f(x,y) \equiv 0$, when $x < 0$, but f is not identically zero on any neighborhood of $(0,0)$. A detailed proof of this example is given together with further references for results on counterexamples.

Definition A.5.8. A function f on a compact K is called *Lipschitz continuous* if $\exists C > 0$ such that $x, y \in K$, $|f(x) - f(y)| < C\|x - y\|$.

When $M \subset \mathbb{C}^n$ is a C^∞-smooth hypersurface and $\gamma \subset M$ a real-analytic curve, then any Lipschitz continuous CR function that vanishes to infinite order along γ, vanishes on an M-open neighborhood of γ. This result is due to Baouendi & Treves [8] (see Treves [72], Theorem II.8.1, together with Corollary II.8.1, p.118, for a textbook version). Let $0 \in \gamma$ and $U \ni 0$ open in M such that $U \cap M = \{y_n = h(z_1, \dots, z_{n-1}, x_n)\}$ for a real-analytic graphing function h. The unique continuation result of Baouendi & Treves [8] in this real-analytic case, is a consequence the so called compact cocycle property (see Treves [72], p.115) for $\{y_n = h(0, x_n)\}$.

The following theorem can be found in Baouendi & Rothschild [10] (see also Shapiro [68]).

Theorem A.5.9. *Let U be an open neighborhood of $x_0 \in \partial B_0(1)$.*
(i) If f is a harmonic function in $U \cap B_0(1)$ and continuous on $\overline{U \cap B_0(1)}$ (where it is assumed that $U \cap B_0(1)$ is connected) such that f vanishes to infinite order in the normal direction at x_0 and $f(x) \geq 0, x \in U \cap \{x \in \mathbb{R}^n : |x| = 1\} = U \cap \partial B_0(1)$, then $f \equiv 0$ on a neighborhood of x_0 in $U \cap \partial B_0(1)$. Furthermore f vanishes identically in the normal direction at x_0 i.e., $f(tx_0) = 0, \forall t \leq 1$ such that the segment $[tx_0, x_0]$ is contained $\overline{U \cap B_0(1)}$.
(ii) If f vanishes to infinite order at x_0, $f(x) \geq 0, x \in U \cap \partial B_0(1)$, then $f \equiv 0$ on $U \cap B_0(1)$.

Definition A.5.10 (see Baouendi & Rothschild [10])**.** Let $U \subset \mathbb{R}^n$ be a neighborhood of some point $x_0 \in \{x \in \mathbb{R}^n : |x| = 1\}$ and set $\Omega = U \cap B_0(1)$. a continuous function, u, is said to vanishing to infinite order if $\lim_{\Omega \ni x \to x_0} \frac{u(x)}{|x - x_0|^N} = 0$. f is said to vanish of infinite order *in the normal direction at* x_0 if $\lim_{t \to 0,\ 0<t<1} \frac{u(tx_0)}{|1-t|^N} = 0$.

These results where followed up, and we should mention the following.

Theorem A.5.11 (Baouendi & Rothschild [9])**.** *If $f(z)$ is a holomorphic function in a domain of the upper half plane with 0 on the boundary, continuous up to the boundary, vanishing to infinite order at 0 and $Ref(x) \geq 0$, $x :=$Rez, then f must vanish identically.*

Theorem A.5.12 (Huang et al. [37]). *If $f = u + iv$ is holomorphic in $H^+ := \{z \in \mathbb{C} : Im z > 0\}$, and continuous up to $(-1, 1) \subset \partial H^+$, such that $|v(t)| \leq |u(t)|$ for $t \in (-1, 1)$, and if f vanishes to infinite order at 0 (in the sense that $f(z) = O(|z|^k)$, for all $k \in \mathbb{N}$), then $f \equiv 0$.*

Remark A.5.13. Let $\omega \subset \mathbb{C}$ be a domain and let f be a function continuous on $\overline{\omega}$ (by which we mean continuous on ω with continuous extension up to the boundary). Assume f vanishes to infinite order at a point $p \in \overline{\omega}$, in the sense of Theorem A.5.12, i.e. $f(z) = O(|z - p|^k)$, $\overline{\omega} \ni z \to p$, $\forall k \in \mathbb{N}$. Note that for any $p \in M$, we have (sufficiently near p), $|f(z)| \cdot |z - p|^{-(k+1)} \leq C_{k+1} \Rightarrow |f(z)| \cdot |z - p|^{-k} \leq C_{k+1} |z - p|$, thus letting $z \to p$, we see that,

$$\lim_{\overline{\omega} \ni z \to p} \frac{f(z)}{|z - p|^k} = 0, \quad k \in \mathbb{N}, \tag{A.32}$$

(where the case $k = 0$ is due to the fact that $|f(z)| \leq C_1 |z - p| \to 0$ as $z \to p$).

Rosay [59] solves a version of our problem is in an elegant manner using a version of the so called jump theorem. The proof also uses a result of Andreotti & Hill [3]. The theorem proved is the following.

Theorem A.5.14. *Let $M \subset \mathbb{C}^n$ be a hypersurface containing the origin, and let γ be the intersection of M with a one-dimensional complex plane L such that L is transverse to $T_0^c M$ (by this we mean that the tangent space of L and $T_0^c M$ generate the ambient tangent space at the origin). If $f \in CR(M)$ such that f vanishes to infinite order on γ (which is formulated as $f(z) = \mathcal{O}(dist(z, \gamma)^k)$, $\forall k \in \mathbb{N}$), then $f \equiv 0$ in a neighborhood of γ in M.*

Definition A.5.15. Let M be a real manifold in a neighborhood of the origin in $\mathbb{C}^n$, $n > 1$, of codimension $d < n$. M is called *generating* if the differentials of the coordinate functions are independent on M, i.e., $dz_1 \wedge \cdots \wedge dz_n|_M \neq 0$. Assume M has defining functions $\rho_1, \ldots, \rho_d$. Then being generating at a point p is equivalent (see Chirka [24], p.105) to $\partial\rho_1 \wedge \cdots \wedge \partial\rho_d|_p \neq 0$. If this equation holds for all sets of local defining functions and for every point then M is generic (see Boggess [21], p.102).

Theorem A.5.16 (Baouendi & Treves [8], Theorem 3.1). *Let $M \subset \mathbb{C}^{n+d}$ be a generic C^1 submanifold of codimension $d \geq 1$, and let γ be a d dimensional submanifold of M, which is equal to the holomorphic-transverse intersection of M with a holomorphic submanifold H of $\mathbb{C}^{n+d}$, H of complex dimension d (by holomorphic-transverse we mean that the tangent space of H at the origin generates together with $T_0^c M$ the ambient tangent space as a direct sum). Assume that M satisfies the following condition, denoted by (A) : Given any open neighborhood W of $0 \in \gamma$ there is*

a holomorphic function F in an open subset of H containing W, such that $F(0) \neq 0$ and the connected component of 0 in the set $\{p \in W : F(p) \neq 0\}$ has compact closure contained in W.
Then if $f \in CR(M)$ is Lipschitz and vanishes to infinite order on γ, then given any open neighborhood U of the origin in M there is an open neighborhood $V \subset U$ of 0 on which $f \equiv 0$.

Theorem A.5.17 (Grachev [31]). *Let $M \subset \mathbb{C}^n$ be a smooth generating CR manifold of codimension $d < n$ containing the origin, and let L be a d dimensional complex plane transverse to $T_0M \cap JT_0M$, and let $\gamma = M \cap L$. If $f \in CR(M)$ and Lipschitz and such that f vanishes to infinite order on γ, which is formulated by saying that as z approaches γ, we have*

$$|f(z)| \leq C_k \left(dist(z,\gamma))^k\right), \quad \forall k \in \mathbb{N}, \tag{A.33}$$

then $f \equiv 0$ in a neighborhood of γ in M.

Hill & Nachinovich [35] prove a unique continuation theorem for CR functions on abstract weakly pseudoconcave CR manifolds (by abstract is meant a formally integrable structure (M, L) with $L_p \cap \overline{L}_p = 0$). Regarding L^1-distributions we have the following theorem, see Berhanu & Hounie [17].

Theorem A.5.18. *Let L be a locally integrable structure defined on a connected open set $\Omega \subset \mathbb{R}^n$. Assume that Ω can be decomposed into $\Omega = \mathfrak{o} \cup Z$, where $\mathfrak{o}$, is an open a.e. minimal orbit of L, and Z is a set of zero measure. Then any solution $u \in L^1_{loc}$ of $Lu = 0$, which vanishes on a set of positive measure, vanishes identically.*

Berhanu & Mendoza [18] prove a uniqueness result for distributions in $L^1_{loc}CR(N)$ where N is an immersed CR submanifold.