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Optimizing pacing strategies on a hilly track in cross-country skiing

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Abstract

During events involving locomotive exercise, such as cross-country skiing, it is believed that pacing strategies (i.e. power distribution) have a significant impact on performance. Therefore, a program was developed for the numerical simulation and optimization of cross-country ski racing, one that numerically computes the optimal pacing strategy for a continuous track. The track is modelled by a set of cubic splines in two dimensions and can be used to simulate a closed loop track or one with the start and finish at different locations. For an arbitrary point on the two dimensional track, equations of motion are formulated parallel and normal to the track, considering the actual slope and curvature of the track. Forces considered at the studied point are the gravitational force, the normal force between snow and skis, the drag force from the wind, the frictional force between snow and ski and the propulsive force from the skier, where the latter is expressed as the available power divided by the actual speed. The differential equations of motion are solved from start to finish using the Runge-Kutta-Fehlberg method. The optimization of the ski race is carried out with the Method of Moving Asymptotes (MMA) which minimizes the racing time by choosing the optimum distribution of available power. Constraints for minimum, maximum and average power are decided by conditions of scaling by body size. Results from a simulated ski competition with optimized power distribution on a real track are presented.

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1. Introduction

Cross-country skiing is a winter sport where the athlete's (skier's) ability to cover the course distance in the shortest time possible is of decisive importance to his/her performance. But since the athlete's amount of power is limited, the efforts must be distributed in a rational way. High power should be used where it is most advantageous, and low power on sections of the track where it is possible or perhaps necessary to rest and recover. Mathematical modelling of a ski race, combined with efficient nonlinear optimization routines, provides the potential to analyze how the skiers' efforts should be distributed in the most advantageous way. Individual differences, such as size, mass, the available power and ability to recover, etc., can be considered, as well as the influence of different course profiles on different skiers. The aim of this investigation was to create a model that can approximate the optimal pacing strategy for a typical world class cross-country sprint skier on a real course profile. And in the long run, the athlete can apply this optimal pacing strategy to improve performance.

Nomenclature

t, n	local coordinate t in the course direction and n normal to the course, aimed at the centre of curvature
x, y	global coordinates horizontally and vertically
$f(x)$	equation for the course section, *i.e.* $y = f(x)$
α	course incline at studied point
R	course curvature at studied point
N	normal force from snow
F_g	gravity force, $F_g = mg$
F_D	drag force (air resistance) also taking into account any tail/headwind w in addition to skiers speed $\dot{t}$
F_μ	frictional force between ski and snow (glide)
F_S	skier's forward propulsive force
w	tail or headwind speed on current course section
P	available (propulsive) power

2. Description of the Numerical Simulation Model

An arbitrary two-dimensional course profile can be created as a connected chain of cubic splines. As the entire course is expressed as a connected chain of third degree polynomials, there is no problem in obtaining the actual gradient and curvature radius for an arbitrary point on the course. The model only takes account of the effective propulsive force generated by the athlete in the direction of the track. This in contrast to Moxnes and Hausken [1] or Holmberg and Lund [2] where certain parts of body movements in skiing were studied. Moxnes and Hausken show analytical results for the relationships between glide length, friction and kicking force during a diagonal stride, and in Holmberg and Lund a simulation of the biomechanics of double-poling was performed and the load distribution between different muscles was analysed.

2.1. Forces and scaling

External forces to take into account during a race are the effect of gravity, the frictional forces between the skis and snow and the effect of air resistance in which any tailwind or headwind can be included. The skier's propulsive force F_S in the direction of the course is used in the equations of motion. The propulsive force is the available power, generated by the skier, divided by the current speed ($F_S = P/v$). When scaling between skiers of different size (mass), it has been assumed that the athletes have uniform physiques. Scaling the terms of $\bar{P}$ and P_{max} in Equations 7-9 for the individual skier was performed using the exponential relationship;

$$\frac{P_1}{P_2} = \left(\frac{m_1}{m_2}\right)^{0.94} \tag{1}$$

where $\bar{P}$ is the allowable maximum value of the mean power and P_{max} is the maximum power for the optimization variables. Index represents skiers 1 and 2 with masses m_1 and m_2 etc., see also Bergh [3]. In the expression used for air resistance, $F_D = \frac{1}{2}C_D A\rho v^2 = kv^2$ the air resistance coefficient k ($k = \frac{1}{2}C_D A\rho$) in uniform scaling is only proportional to the projected frontal area A and is therefore scaled according to Equation 2, as the projected frontal areas behave like masses raised to 2/3.

$$\frac{k_1}{k_2} = \left(\frac{m_1}{m_2}\right)^{0.67} \tag{2}$$

2.2. Derivation of equations of motion for an arbitrary course section

The equations of motion are drawn up for a skier in horizontal and vertical directions for an arbitrary point on the course, see Figure 1. Speed and acceleration relationships in Equation 3 and geometric relationships from the course profile equation in Equation 4:

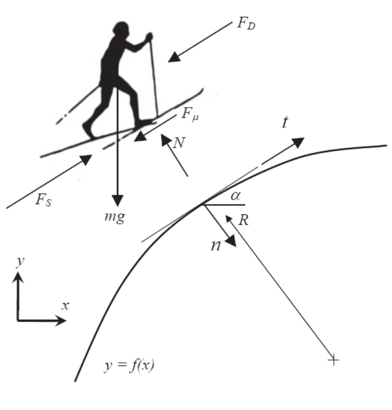

Fig. 1. Arbitrary section of the course with forces on the skier and chosen coordinates.

$$\dot{t} = \sqrt{\dot{x}^2 + \dot{y}^2} \qquad \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} \cdot \begin{bmatrix} \ddot{t} \\ \ddot{n} \end{bmatrix} \tag{3}$$

$$\alpha = \arctan(y') \qquad \frac{1}{R} = \frac{y''}{[1+(y')^2]^{3/2}} \tag{4}$$

During a ski race a skier is obliged to follow the given course profile. As a first step, it is thus useful to describe the equations of motion in the natural directions of the movement, *i.e.* in the direction normal to the course and tangentially [4]. After transformation to *x*- and *y*-coordinates and inserting the expressions of the acting forces, the following equations of motion are obtained:

$$m\ddot{x} = \frac{P}{\dot{t}}\cos\alpha - k(\dot{t}+w)^2\cos\alpha - m\left(g\cos\alpha - \frac{\dot{t}^2}{R}\right)(\mu\cos\alpha + \sin\alpha),$$

$$m\ddot{y} = \frac{P}{\dot{t}}\sin\alpha - k(\dot{t}+w)^2\sin\alpha - m\left(g\cos\alpha - \frac{\dot{t}^2}{R}\right)(\mu\sin\alpha + \cos\alpha) - mg. \tag{5}$$

where the frictional force is $F_\mu = \mu m(g\cos\alpha - \frac{\dot{t}^2}{R})$, the propulsive force is $F_s = \frac{P}{\dot{t}}$ and the drag force is $F_D = \frac{1}{2}C_D A\rho(\dot{t}+w)^2 = k(\dot{t}+w)^2$. In the drag force expression $k = \frac{1}{2}C_D A\rho$, C_d is the drag coefficient, A is projected frontal area, ρ is air density and w is the tailwind or headwind speed. The second order differential equations in Equation 5 can be transformed into a system of four connected first order equations. Subsequently the system may be solved by any standard numerical solver, see Carlsson *et al.* [5].

3. Optimization

Numerical optimization is performed on the previously described simulation program. The natural choice of objective function in the optimization is to minimize the total race time (Equation 6). The chosen optimization variables are available power on carefully selected points along the course (P_j). Power levels between these points are available from linear interpolation between the variables. Since the maximum power a skier can generate over a longer time is limited, certain restrictions must be connected to the objective function. The chosen constraints are maximum and minimum levels of power (Equation 7), combined with the global maximum mean power $\bar{P}$ during the whole race (Equation 8). In order to prevent the skier from using excessively high power on a longer part of the course, local constraints are added to those previously described (Equation 9). These constraints prevent the skier from developing maximum power on more than two consecutive optimization variables and allow a period of recovery before the next maximum outtake. In mathematical terms, the optimization problem is formulated as;

$$\text{Minimize} \quad T = \sum_{i=1}^{K} \Delta t_i \tag{6}$$

$$\text{Such that} \quad P_{min} \le P_j \le P_{max} \qquad j = 1, 2, \ldots, N \tag{7}$$

$$\frac{1}{T}\int_0^T P(t)dt \le \bar{P} \tag{8}$$

$$\left(P_j + P_{j+1} + P_{j+2}\right) \le 2P_{max} + C \cdot \bar{P} \qquad j = 1, 2, \ldots, (N-2) \tag{9}$$

where T is the total race time, Δt_i is the time segment during iteration i, K is the total number of time segments during the actual simulation, P_j is the *j:th* optimization variable (*i.e.* power level at j), P_{min} and P_{max} are the minimum and maximum power levels for the optimization variables, $\bar{P}$ is a reference value of the maximum mean power for the simulated skier, C is a constant ($0 \le C \le 1$) and N is the number of optimization variables. To ensure that a good speed is maintained at the very end of the race, it might sometimes be necessary to complete the constraints with restrictions to certain speeds or power levels for the last variables.

4. Numerical Results

The transformed system of connected first order differential equations of movement has been implemented in a MATLAB program for solutions with the Runge-Kutta-Fehlberg method. In this paper, the gradient-based Method of Moving Asymptotes (MMA [6]) has been used for the optimization. The routine has performed excellently in a number of optimization applications, including structural dynamics [7].

Experience shows that it is important to keep the time segments small when solving the differential equations, otherwise the optimization variables (*i.e.* the power variables) will occur on slightly different places on the track with different levels of power. The input data for solving the differential equations are the skier's mass, starting speed and available power during the various parts of the race. Input data also includes the course profile expressed as a connected chain of cubic splines, the glide friction coefficient μ, tailwind or headwind speed w and any different postures' effect on air resistance on various sections of the course. When scaling the power constraints ($\bar{P}$, P_{max}) and the coefficient k in the expression of air resistance, the reference value has been taken from values that may be appropriate for a skier with the mass m = 70 kg. For this skier the reference value of maximum mean power ($\bar{P}$) has been set to $\bar{P}_{70}$ = 325 W and the air resistance coefficient k_{70} = 0.49. The reference values used for power, air resistance and friction have, when simulating skiing 10,000 m on a flat course in conditions with no wind, given a skiing time of just under 24 min. for a skier with a mass of 70 kg.

The planned course for the Swedish Sprint Championships in 2007 has been modelled as a chain of totally 36 connected cubic splines, see Figure 2 (the sprint competitions were actually moved to another place due to a lack of snow in the intended area). The course has a length of 1425 m. The course has been used for the optimization of a skier's power distribution during the race. In the simulated race a friction coefficient of μ = 0.03 is used and no headwind or tailwind is assumed. Optimization is performed with N = 35 variables. The power variables are spread over the course in such a way that they are closer to each other where the slope of the course has distinct changes. Available power values between the variable values have been calculated by linear interpolation. Maximum and minimum values for the power in Equation 8 are set to $P_{min} = 0 \cdot \bar{P}$ resp. $P_{max} = 1.2 \cdot \bar{P}$ and the constant C in Equation 9 is given the value C = 1. In order to stabilize the numerical process, all constraints are normalized during the optimization.

4.1. Optimization of a skier on the planned course for the Swedish Sprint Championships in 2007

Considering a typical world class cross-country sprint skier with a body mass of 75 kg, we obtain $\bar{P}$ = 346.8 W and k = 0.51. The maximum and minimum values for the power variables are P_{min} = 0.0 W and P_{max} = 416.1 W. The constraints on the two last variables remain the same and optimization starts with initial values of $P_j = \bar{P}$ = 346.8 W for all variables P_j. This initial set of variables gives a total time of 3 min 27.9 s with all constraints fulfilled. After 15 iterations with the optimization routine, the total time has decreased to 3 min 24.55 s, a time difference of -3.55 s. All the constraints are still fulfilled and the divergence from the stipulated reference value of the mean power in Equation 8 is now less than 0.5 W.

The location of the power variables (P_j) and their values after optimization are shown in Figure 2. The mean power that the athlete is able to sustain for a longer period of time is shown as a dashed line in the figure. Because of the formulation of the constraint in Equation 8, the area between the line for the power distribution should be equal above and below the mean power level (provided that the constraint is fulfilled with equality).

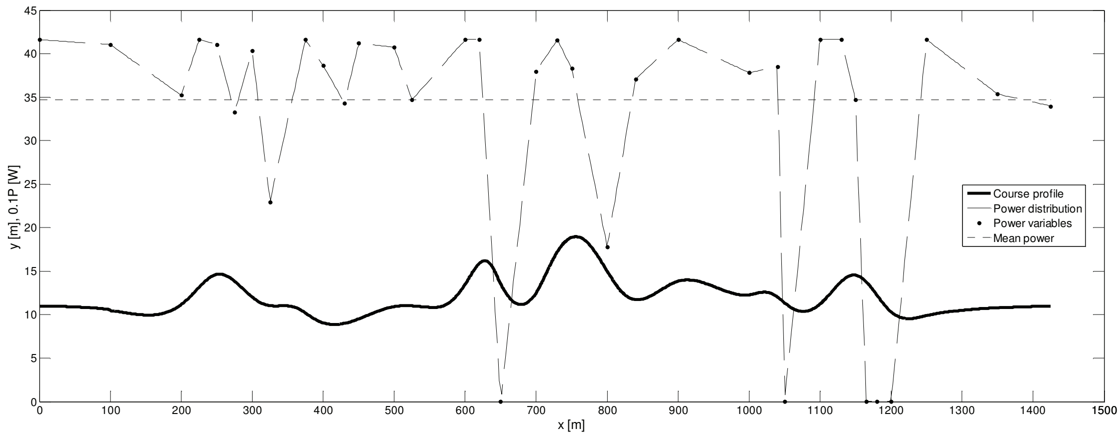

Fig. 2. Course profile and optimized power distribution for a 75 kg skier.

5. Discussion and Conclusion

The planned course for the Swedish Sprint Championships in 2007 was modelled with a chain of cubic splines. Skiing was simulated for a skier of 75 kg, and optimization was performed on the power distribution along the track. The main goal of the optimization was to minimize the total race time, at the same time as there were limitations to the available power during the race. Overall, the performed optimization acts in a reliable way. From an evenly distributed power level, the optimization routine automatically iterates to a more effective way of distributing effort along the course. The risk of fatigue for the skier can be avoided by constraints which control power levels during the race. A global constraint on the power level is necessary but not enough; even if the mean level of the power is below that prescribed there is still a risk of connected passages with excessive power output. The local constraint in Equation 9 for connected power outtake assures that this risk is eliminated.

Individual adaption of optimization is easy; different bodily constitutions, as well as different physical capacities, can be modelled. If necessary, the optimization problem can be completed with more constraints to make it work better. A skier's ability to perform their very best can of course vary from day to day and this performance "on the day" can have a great effect on the results. The simulation shown here does not take such variations into account, but it is nonetheless of great interest to study how power should be distributed in the most effective way during a ski race. Once the profile of a course is known, such studies might serve as an essential part of the preparation for important championship races.

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