Des. Codes Cryptogr. (2014) 71:247–260
DOI 10.1007/s10623-012-9730-2

Inner balance of symmetric designs

Tomas Nilson · Pia Heidtmann

Received: 21 October 2011 / Revised: 8 May 2012 / Accepted: 10 July 2012 /
Published online: 22 July 2012

Abstract A *triple array* is a row-column design which carries two *balanced incomplete block designs* (BIBDs) as substructures. McSorley et al. (Des Codes Cryptogr 35: 21–45, 2005), Section 8, gave one example of a triple array that also carries a third BIBD, formed by its row-column intersections. This triple array was said to be *balanced for intersection*, and they made a search for more such triple arrays among all potential parameter sets up to some limit. No more examples were found, but some candidates with suitable parameters were suggested. We define the notion of an *inner design with respect to a block* for a symmetric BIBD and present criteria for when this inner design can be balanced. As triple arrays in the canonical case correspond to SBIBDs, this in turn yields new existence criteria for triple arrays balanced for intersection. In particular, we prove that the residual design of the related SBIBD with respect to the defining block must be quasi-symmetric, and give necessary and sufficient conditions on the intersection numbers. This, together with our parameter bounds enable us to exclude the suggested triple array candidates in McSorley et al. (Des Codes Cryptogr 35: 21–45, 2005) and many others in a wide search. Further we investigate the existence of SBIBDs whose inner designs are balanced with respect to every block. We show as a key result that such SBIBDs must possess the *quasi-3* property, and we answer the existence question for all known classes of these designs.

Keywords Symmetric design · Triple array · Balanced for intersection · Quasi-3 design · Inner design with respect to a block · Quasi-symmetric design

Mathematics Subject Classification (2010) 05B05 · 05B30

Communicated by J. D. Key.

T. Nilson (✉) · P. Heidtmann
Department of Applied Science and Design, Mid Sweden University,
851 70 Sundsvall, Sweden
e-mail: tomas.nilson@miun.se

P. Heidtmann
e-mail: Pia.Heidtmann@miun.se

1 Introduction

We assume familiarity with the basic terminology of designs, but all preliminary definitions and results needed can be found in "Appendix".

One reason to study designs is the many applications in statistical experiments, and these have influenced the properties we choose to investigate. One such important property when considering row-column designs is *adjusted orthogonality*, which was defined in Eccleston and Russell [9], but the combinatorial property had already been investigated by researchers several years prior to that. Agrawal [1] studied such a class of designs used for *two-way elimination of heterogeneity* that later got the name *triple arrays*, and these designs have recently been studied by McSorley et al. [14] among others. The following two definitions are equivalent and both are used.

Definition 1.1 Let $\mathcal{A}$ be an $r \times c$ row-column design on v symbols that satisfies

(1) the rows are the dual of a BIBD,
(2) the columns are the dual of a BIBD,
(3) every row intersects every column in a constant number of symbols,

then $\mathcal{A}$ is called a *triple array*.

Definition 1.2 Let $\mathcal{A}$ be a binary $r \times c$ row-column design on v symbols, equireplicate with replication number k, where $k < r, c$, and let $\lambda_{rr}, \lambda_{cc}$ and λ_{rc} be positive integers. If $\mathcal{A}$ satisfies that

(1) any two distinct rows intersect in λ_{rr} symbols;
(2) any two distinct columns intersect in λ_{cc} symbols;
(3) any row and column intersect in λ_{rc} symbols;

then $\mathcal{A}$ is called a *triple array*, denoted by $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$.

The third property in Definitions 1.1 and 1.2 is known as the adjusted orthogonality property for row-column designs. Since $\mathcal{A}$ is equireplicate it means that the row-column intersections in a triple array form a block design. McSorley et al. [14] observed that the row-column intersections of a $TA(10, 3, 3, 2, 3 : 5 \times 6)$ form a *balanced incomplete block design* (BIBD), with parameters $(10, 30, 9, 3, 2)$ and called this a *triple array balanced for intersection*, asking if there are more of this kind. They made a computer search up to $r = 100$. No more balanced examples were found, but the search gave some potential candidates with suitable parameters. These potential candidates were stated both in terms of triple array parameters and in terms of parameters for the related *symmetric incomplete block design* (SBIBD) which in the canonical case is known to exist for every triple array (a proof of this can be found in [14]).

Theorem 1.3 *Suppose $\mathcal{A}$ is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with $v = r + c - 1$. Then there exists a $(v + 1, r, \lambda_{cc})$-SBIBD.*

McSorley et al. [14] proved that any $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ satisfies that $v \geq r + c - 1$. They also presented the only known example with $v > r + c - 1$. All other known triple arrays satisfy $v = r + c - 1$ and thus correspond to SBIBDs. Because of this, strong results on non-existence of SBIBDs could be used to eliminate some of the potential triple arrays. Note that the converse of Theorem 1.3 is open, and is known as *Agrawal's Conjecture*.

Conjecture 1.4 [1] *If there is a $(v + 1, r, \lambda_{cc})$-SBIBD with $r - \lambda_{cc} \geq 2$ then there is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$ with $v = r + c - 1$.*

In this paper we improve on the strategy of McSorley et al. [14]. In Sect. 2 we define the *inner design* with respect to the blocks of an SBIBD, such that this inner design corresponds to the row-column intersections of a triple array, when a triple array exists. We develop criteria for when the inner design is a BIBD, in which case we say that the SBIBD has *inner balance*. These results with their roots in the much more well-studied theory for SBIBDs are then subsequently used to characterize SBIBDs with inner balance. This enables us to exclude the suggested triple array candidates in [14] and many others in an enhanced computer search.

Because the inner design is defined with respect to a particular block in the SBIBD, the natural question arises as to when it can be defined with respect to every block. In Sect. 3 we prove that this can happen only if the SBIBD is *quasi-3*. We thus go on to answer the existence question for SBIBDs with inner balance for all known parameter classes of quasi-3 designs.

2 Results

Given a (v, k, λ)-SBIBD, we want to define a block design in the SBIBD that will correspond to the row-column intersection design of a triple array.

Proposition 2.1 *Let $\mathcal{D} = (\mathcal{X}, \mathcal{B})$ be a (v, k, λ)-SBIBD and let $B_0 \in \mathcal{B}$ be a fixed block. For s such that $B_s \in \mathcal{B} \setminus \{B_0\}$, i such that $x_i \in B_0$, and j such that $x_j \in X \setminus B_0$, let*

$$R_i = \{s : x_i \in B_0 \setminus B_s\}; \quad C_j = \{s : x_j \in B_s \setminus B_0\}.$$

Then the sets $R_i \cap C_j$ form the blocks of an incomplete block design with parameters

$$\left(v - 1, k(v - k), (k - \lambda)^2, k - \lambda\right),$$

which we call the inner design of $\mathcal{D}$ with respect to B_0 *and denote by $\mathcal{D}_\star$.*

Proof The point set of $\mathcal{D}_\star$ is the set of block indices of $\mathcal{D}$ except 0, let us call it $S = \{1, 2, \ldots, v - 1\}$. The number of blocks is $k(v - k)$ because every pair (i, j), where $x_i \in B_0$, $x_j \in X \setminus B_0$ gives a set $R_i \cap C_j$. As $\mathcal{D}$ is an SBIBD, we know that for a fixed pair (i, j), that $|R_i \cap C_j| + |(S \setminus R_i) \cap C_j| = k$ and $|(S \setminus R_i) \cap C_j| = \lambda$, so the block size of $\mathcal{D}_\star$ is constant and is given by $|R_i \cap C_j| = k - \lambda$. An element $s \in S$ occurs once in $D_\star$ for each pair (i, j) with $x_i \in B_0 \setminus B_s$, $x_j \in B_s \setminus B_0$. Hence every s occurs exactly $|B_0 \setminus B_s| \cdot |B_s \setminus B_0| = (k - \lambda)^2$. As $k < v$ and $\lambda \geq 1$ we have that $k - \lambda < v - 1$, so $\mathcal{D}_\star$ is incomplete. □

Proposition 2.2 *Let $\mathcal{D}$ be a (v, k, λ)-SBIBD and let $\mathcal{D}_\star$ be the inner design of $\mathcal{D}$ with respect to a block B_0. If $\mathcal{D}_\star$ is balanced, then its balance index $\lambda_\star$ is given by*

$$\lambda_\star = \frac{\lambda(k - \lambda)^2(k - \lambda - 1)}{k^2 - k - \lambda},$$

and we say that $\mathcal{D}$ has inner balance with respect to B_0.

Proof Suppose $\mathcal{D}_\star$ is balanced. Then we can calculate its balance index $\lambda_\star$ using the fundamental identities for BIBDs in Proposition A.2

$$\lambda_\star = \frac{(k - \lambda)^2(k - \lambda - 1)}{v - 2}. \tag{2.1}$$

As the replication number and block size are equal in an SBIBD, we can reuse one of the fundamental identities to get

$$v - 2 = \frac{k(k-1)}{\lambda} - 1 = \frac{k^2 - k - \lambda}{\lambda}.$$

Substituting into the right side of Eq. 2.1 gives the result. □

The construction of an SBIBD $\mathcal{D}$ from a triple array implies that the dual of the rows in the triple array is the complementary design of the derived design modulo a block B in $\mathcal{D}$ (cf. [14]). Similarly, the dual of the columns is the residual design modulo B in $\mathcal{D}$. We note that these are exactly the sets R_i and C_j respectively forming the blocks of the design in Proposition 2.1, so the row-column intersections of the triple array become the sets $R_i \cap C_j$, and we get the following lemma.

Lemma 2.3 *Let $\mathcal{A}$ be a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with $v = r + c - 1$. Then the block design formed by its row-column intersections is isomorphic to an inner design $\mathcal{D}_\star$ of the corresponding $(v + 1, r, \lambda_{cc})$-SBIBD.*

Note that it is also possible to define an inner design in a dual manner. Let $\mathcal{D} = (X, \mathcal{B})$ be a (v, k, λ)-SBIBD and let $x_0 \in X$ be a fixed point. For any block B_i containing x_0 and any block B_j not containing x_0, define $R_i = X \setminus B_i$ and $C_j = B_j$. Then the sets $R_i \cap C_j$ form the blocks of an incomplete block design with point set $X \setminus \{x_0\}$ and parameters $\big(v - 1, k(v - k), (k - \lambda)^2, k - \lambda\big)$, which we call the inner design of $\mathcal{D}$ with respect to the point x_0. This definition coincides with the definition of Proposition 2.1 in the case when $\mathcal{D}$ is self-dual. In this paper we do not merely consider SBIBDs, but have the additional requirement that row-column intersections of a triple array should correspond to the blocks of the inner design, hence the slightly more cumbersome definition of inner balance with respect to a block is more natural in our case.

The complementary design of an SBIBD is also an SBIBD, and the next proposition shows that the two designs possess the same inner designs.

Proposition 2.4 *Let $\mathcal{D} = (X, \mathcal{B})$ be an SBIBD and let $\mathcal{D}'$ be its complementary design. Then the inner design of $\mathcal{D}$ with respect to a block B_0 is the same as the inner design of $\mathcal{D}'$ with respect to $B'_0 := X \setminus B_0$.*

Proof The point sets of $\mathcal{D}'_\star$ and $\mathcal{D}_\star$ are the same set of block indices of $\mathcal{D}$. The blocks of $\mathcal{D}'_\star$ are the pairwise intersections of the sets $R_i = \{s : x_i \in (X \setminus B_0) \setminus (X \setminus B_s)\}$ and the sets $C_j = \{s : x_j \in (X \setminus B_s) \setminus (X \setminus B_0)\}$. These form the same blocks as in $\mathcal{D}_\star$ since

$$(X \setminus B_0) \setminus (X \setminus B_s) = (X \setminus B_0) \cap B_s = B_s \setminus B_0$$

and

$$(X \setminus B_s) \setminus (X \setminus B_0) = (X \setminus B_s) \cap B_0 = B_0 \setminus B_s.$$ □

Because complementary designs have the same inner design, we thus often consider only those with the smaller block size. The (11, 5, 2)-SBIBD is unique up to isomorphism (cf. [6]) and has inner balance (cf. [14]). Consequently the (11, 6, 3)-SBIBD also has inner balance, as it is the complementary design of the (11, 5, 2)-SBIBD. In Sect. 3 we will discuss the choice of defining block for the inner design. We shall see that in the (11, 5, 2)-SBIBD case, the inner design can be defined with respect to every block and is a (10, 30, 9, 3, 2)-BIBD.

Lemma 2.5 *Let $\mathcal{D} = (X, \mathcal{B})$ be a (v, k, λ)-SBIBD and let $B_0 \in \mathcal{B}$. Suppose that the residual design $\mathcal{D}^{B_0}$ has n intersection numbers, say $0 \leq \mu_{R_0} < \mu_{R_1} < \cdots < \mu_{R_{(n-1)}}$. Then the derived design $\mathcal{D}_{B_0}$ has n intersection numbers too, say $0 \leq \mu_{D_0} < \mu_{D_1} < \cdots < \mu_{D_{(n-1)}}$, and we have that $\mu_{R_\alpha} + \mu_{D_{(n-1-\alpha)}} = \lambda$ for all $\alpha \in \{0, 1, \ldots, n-1\}$.*

Proof By Proposition A.11 we know that an SBIBD $\mathcal{D}$ has only one intersection number $\mu = \lambda$, whereas a non-symmetric design like $\mathcal{D}^{B_0}$ has at least two. Taking subdesigns $\mathcal{D}^{B_0}$ and $\mathcal{D}_{B_0}$ of $\mathcal{D}$ imply a partition of every block $B_s \in \mathcal{B}\setminus\{B_0\}$ into $B_s \setminus B_0$ and $B_0 \cap B_s$ respectively. So if two distinct blocks $B_s, B_t \in \mathcal{B}\setminus\{B_0\}$ intersect in μ_{R_α} points in $\mathcal{D}^{B_0}$, then the remaining $\lambda - \mu_{R_\alpha}$ intersections happen in $\mathcal{D}_{B_0}$. So for each μ_{R_α}, where $\mu_{R_0} < \mu_{R_1} < \cdots < \mu_{R_{(n-1)}}$, there is a corresponding $\mu_{D_{(n-1-\alpha)}}$, where $\mu_{D_0} < \mu_{D_1} < \cdots < \mu_{D_{(n-1)}}$ satisfying the required sum. □

Lemma 2.6 *Let $\mathcal{D} = (X, \mathcal{B})$ be a (v, k, λ)-SBIBD and let B_0, B_s, B_t be three distinct blocks in $\mathcal{B}$. Then $|(B_s \cap B_t) \setminus B_0|$ is an intersection number of the residual design $\mathcal{D}^{B_0}$, say μ_{R_α}. Further the sets R_i and C_j defined in Proposition 2.1 with respect to B_0 satisfy the following*

(1) the number of sets C_j in which both s and t occur is μ_{R_α};
(2) the number of sets R_i in which both s and t occur is $k - \lambda - \mu_{R_\alpha}$;
(3) the number of sets $R_i \cap C_j$ in which both s and t occur is $(k - \lambda)\mu_{R_\alpha} - \mu_{R_\alpha}^2$.

Proof Clearly, the symbols s and t both occur in C_j when $x_j \in ((B_s \cap B_t) \setminus B_0)$. Likewise, the number of sets R_i where both s and t occur is an intersection number of the complementary design of the derived design $\mathcal{D}_{B_0}$. These are given by the Sieve Principle to be $k - 2\lambda + \mu_{D_{(n-1-\alpha)}}$, which we can write as $k - \lambda - \mu_{R_\alpha}$ by Lemma 2.5. Finally, we get the number of sets $R_i \cap C_j$ where both s and t occur from the product $\mu_{R_\alpha}(k - \lambda - \mu_{R_\alpha})$. □

Corollary 2.7 *Let $\mathcal{D}$ be an SBIBD with inner balance with respect to a block B_0, and let μ_R and μ_D be intersection numbers of $\mathcal{D}^{B_0}$ and $\mathcal{D}_{B_0}$ respectively, then*

$$\mu_R \geq 1 \quad \textit{and} \quad \mu_D \geq 0.$$

Proof That $\mu_R \geq 1$ follows directly from (3) in Lemma 2.6, and since it is well known that the residual design of the $(11, 5, 2)$-SBIBD with respect to any block has intersection numbers $\mu_{R_0} = 1$ and $\mu_{R_1} = 2$, we know by Lemma 2.5 that both μ_R and μ_D can attain these bounds. □

Here follows one of our main results which gives necessary and sufficient conditions on the intersection numbers of the residual design.

Theorem 2.8 *Let $\mathcal{D}$ be a (v, k, λ)-SBIBD with $k - \lambda > 1$. Then $\mathcal{D}$ has inner balance with respect to a block B_0 if and only if the residual design $\mathcal{D}^{B_0}$ is quasi-symmetric with intersection numbers that are roots of the equation*

$$x^2 - (k - \lambda)x + \frac{\lambda(k - \lambda)^2(k - \lambda - 1)}{k^2 - k - \lambda} = 0.$$

Proof Suppose $\mathcal{D}$ has inner balance with respect to a block B_0. From property (3) of Lemma 2.6 we get the equation $\mu_{R_\alpha}^2 - (k - \lambda)\mu_{R_\alpha} + \lambda_\star = 0$. Because $\lambda_\star$ is fixed, this equation defines at most two possible values of μ_{R_α}, and since $\mathcal{D}^{B_0}$ is non-symmetric we know by Proposition A.11 that $\mathcal{D}^{B_0}$ has at least two intersection numbers. So the equation has exactly two roots, say μ_{R_0} and μ_{R_1}, and $\mathcal{D}^{B_0}$ is quasi-symmetric. Further, by Proposition 2.2

we can express $\lambda_\star$ in the required way. Conversely, suppose that $\mathcal{D}^{B_0}$ is quasi-symmetric with intersection numbers μ_{R_0} and μ_{R_1} that are roots of the quadratic equation in the theorem. Then μ_{R_0}, μ_{R_1} satisfy $\lambda_\star = (k-\lambda)\mu_{R_\alpha} - \mu_{R_\alpha}^2$ of Lemma 2.6, where $\lambda_\star$ is the balance index of the inner design. The condition $k-\lambda > 1$ guarantees that μ_{R_0} and μ_{R_1} are nonzero, and as they are distinct, this renders $\lambda_\star$ positive. □

By the well-known relations between roots and coefficients of quadratic equations we have the following.

Corollary 2.9 *Let $\mathcal{D}$ be a (v,k,λ)-SBIBD with inner balance with respect to a block B_0, and let μ_{R_0} and μ_{R_1} be the intersection numbers of $\mathcal{D}^{B_0}$. Then*

$$\lambda_\star = \mu_{R_0}\mu_{R_1} \quad \textit{and} \quad k-\lambda = \mu_{R_0} + \mu_{R_1}.$$

Proposition 2.10 *Let $\mathcal{D}$ be a (v,k,λ)-SBIBD with inner balance with respect to a block B_0, then*

$$\lambda + 3 \le k \le 3\lambda - 1.$$

Proof From Theorem 2.8 we know that $\mathcal{D}^{B_0}$ and $\mathcal{D}_{B_0}$ are quasi-symmetric, and from Corollary 2.7 we know that $1 \le \mu_{R_0} < \mu_{R_1}$ and $0 \le \mu_{D_0} < \mu_{D_1}$. This tells us that $\mu_{R_0} + \mu_{R_1} \ge 3$ and $\mu_{D_0} + \mu_{D_1} \ge 1$. By Corollary 2.9 we then have $k-\lambda = \mu_{R_0} + \mu_{R_1} \ge 3$, which gives us the lower bound. Then we use Lemma 2.5 to write

$$k-\lambda = \mu_{R_0} + \mu_{R_1} = \lambda - \mu_{D_1} + \lambda - \mu_{D_0} = 2\lambda - (\mu_{D_1} + \mu_{D_0})$$

so $3\lambda - k = \mu_{D_1} + \mu_{D_0} \ge 1$, which gives us the upper bound. □

Corollary 2.11 *Let $\mathcal{D}$ be a (v,k,λ)-SBIBD with inner balance and $\lambda \le 2$, then $\mathcal{D}$ is the unique $(11,5,2)$-SBIBD.*

Proof Inserting $\lambda = 1$ in the inequality of Proposition 2.10 immediately gives something false. Inserting $\lambda = 2$ gives $k = 5$, and using Proposition A.2 we can calculate $v = 11$. We know that the $(11,5,2)$-SBIBD is unique up to isomorphism (cf. [6]), and we know that it has inner balance (cf. [14]). □

As the $(11,5,2)$-SBIBD is a Hadamard 2-design, this naturally opens the question as to whether this class of designs all have inner balance. We conclude this part of the section by answering this question in the negative.

Definition 2.12 Let $n \ge 2$ be a positive integer. A $(4n-1, 2n-1, n-1)$-SBIBD is called a *Hadamard* 2*-design* of order n.

Proposition 2.13 *Let $\mathcal{D}$ be a Hadamard 2-design with inner balance. Then $\mathcal{D}$ is the unique $(11,5,2)$-SBIBD.*

Proof Inserting $v = 4n-1$, $k = 2n-1$ and $\lambda = n-1$ in the quadratic equation of Theorem 2.8 gives

$$x^2 - nx + \frac{n^2(n-1)}{4n-3} = 0.$$

For the roots to be integers, easy computations show that $\sqrt{4n-3}$ must divide n, say $n = c\sqrt{4n-3}$ for some $c \in \mathbb{Z}_+$. Then $n^2 = c^2(4n-3)$ and

$$n^2 - 4c^2 n + 3c^2 = 0. \tag{2.2}$$

Solving for n gives $n = 2c^2 \pm c\sqrt{4c^2-3}$, so $\sqrt{4c^2-3} = m$ for some $m \in \mathbb{Z}_+$. Squaring this yields $4c^2 - 3 = m^2$ which can be written as $(2c-m)(2c+m) = 3$. As 3 is a prime we have that $c = 1$ and Eq. 2.2 becomes

$$n^2 - 4n + 3 = 0,$$

which has roots 1 and 3. □

2.1 A search for possible parameters for SBIBDs with inner balance

McSorley et al. [14] made a computer search for symmetric designs with inner balance. The search was relevant for $\lambda \leq 97$. Except for the $(11, 5, 2)$-SBIBD, no examples were found, but a couple of potential parameter sets for candidates were identified. An easy calculation eliminates these candidates by our results above combined with the following two. The first gives necessary conditions for quasi-symmetry, the second is the well-known Bruck-Ryser-Chowla Theorem and their proofs can be found in [12].

Proposition 2.14 [10,17]. *If μ_0 and μ_1 are the intersection numbers of a quasi-symmetric (v, b, r, k, λ)-design, then $\mu_1 - \mu_0$ divides both $k - \mu_0$ and $r - \lambda$.*

Theorem 2.15 (*BRC*). *Suppose there exists a symmetric balanced incomplete block design with parameters (v, k, λ).*

(1) If v is even, then $k - \lambda$ is a perfect square;
(2) If v is odd, then there are integers x, y, and z, not all zero, such that

$$x^2 = (k-\lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2.$$

We summarize by defining our candidate set Ω of potential parameter sets for SBIBDs with inner balance.

Definition 2.16 Let $\Omega \subseteq \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+$ be the set of ordered triples (v, k, λ) that satisfy Proposition A.2, Theorem 2.8 within bounds given by Proposition 2.10 such that Proposition 2.14 and Theorem 2.15 are satisfied.

Note that if the necessary conditions of Theorem 2.8 are satisfied, it also means that the divisibility condition of Proposition 2.2 is satisfied.

We will have use of the following result, the content of which can be found in most text books in number theory, e.g. Rosen [16].

Proposition 2.17 *If d is a square-free integer, then the Pell equation $x^2 - dy^2 = 1$ has infinitely many positive integer solutions. Further, if (x_1, y_1) is the least positive solution and (x_n, y_n) is one of the positive solutions, then*

$$x_{n+1} = x_1 x_n + d y_1 y_n$$
$$y_{n+1} = y_1 x_n + x_1 y_n$$

Now we are ready to present an infinite family of parameter sets for potential SBIBDs with inner balance.

Proposition 2.18 *Given any solution (s, t) of the Pell equation $s^2 - 3t^2 = 1$ for positive integers s and t with $s > 2$ and t odd. Take $v = 4t^2 + 2$, $k = 2t^2 - s + 1$, $\lambda = t^2 - s + 1$, then the triple (v, k, λ) is in the candidate set Ω of Definition 2.16.*

Proof The least positive solution of the Pell equation is $(s_1, t_1) = (2, 1)$, and the recursive formulas of Proposition 2.17 give that the first interesting solution is $(s_3, t_3) = (26, 15)$. Hence, we need only consider $s \geq 26$ and $t \geq 15$. Next, we check that the fundamental identity (2) of Proposition A.2 is satisfied.

$$\lambda(v-1) = (t^2 - s + 1)(4t^2 + 1) = 4t^4 + 5t^2 - 4t^2 s - s + 1$$
$$k(k-1) = (2t^2 - s + 1)(2t^2 - s) = 4t^4 + 5t^2 - 4t^2 s - s + 1.$$

After checking that $k - \lambda = t^2 > 1$ we solve the equation of Theorem 2.8

$$x^2 - t^2 x + \frac{t^2(t^2-1)}{4} = 0,$$

which has the integer roots $\frac{t(t-1)}{2}$ and $\frac{t(t+1)}{2}$. For the bounds of Proposition 2.10 we have the left inequality $t^2 - s + 4 \leq 2t^2 - s + 1$, which is true for all integers $t \geq 2$. The right inequality can, after simplification, and using the Pell equation be written as $s^2 - 6s + 2 \geq 0$, which is true for all integers $s \geq 6$. The two divisibility conditions of Proposition 2.14 must now be checked. From Proposition A.6 we have that the parameters of the residual design are

$$(2t^2 + s + 1, 4t^2 + 1, 2t^2 - s + 1, t^2, t^2 - s + 1).$$

The first divisibility condition is

$$\frac{t^2 - \frac{t(t-1)}{2}}{\frac{t(t+1)}{2} - \frac{t(t-1)}{2}} = \frac{t+1}{2},$$

which is an integer if and only if t is odd. The second condition is also satisfied as

$$\frac{2t^2 - s + 1 - (t^2 - s + 1)}{\frac{t(t+1)}{2} - \frac{t(t-1)}{2}} = \frac{t^2}{t} = t.$$

Finally, the BRC Theorem 2.15 holds as $v = 4t^2 + 2$ is even and $k - \lambda = t^2$. □

Example 2.19 Using the recurrence relation of Proposition 2.17 *we compute the first two solutions of the Pell equation in Proposition* 2.18 *with* $s > 2$ *and* t *odd to be* $(s_3, t_3) = (26, 15)$ *and* $(s_5, t_5) = (362, 209)$. *Using Proposition* 2.18 *yields the parameter sets for the corresponding potential SBIBDs. We get*

$$(902, 425, 200) \quad \textit{and} \quad (174726, 87001, 43320).$$

Note that it is not known whether they exist or have inner balance, but these triples lie in the candidate set Ω.

An enhanced computer search for potential parameter sets in Ω for all $\lambda \leq 10^6$ found that, apart from their complementary designs, the $(11, 5, 2)$-SBIBD and the two potential candidates of Example 2.19 are the only three possible SBIBDs with $\lambda \leq 10^6$ that can have inner balance.

Let us for a moment return to the original question about triple arrays. By Theorem 1.3 we know that given a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with $v = r + c - 1$, there is a $(v+1, r, \lambda_{cc})$-SBIBD. We also know that the block design formed by the row-column intersections of the triple array is isomorphic to the inner design of the corresponding SBIBD w.r.t. some block.

Proposition 2.18, using the appropriate solutions (s, t) of its Pell equation suggests an infinite family of parameter sets for potential triple arrays balanced for intersection with parameters

$$TA\left(4t^2 + 1, t^2, t^2 + s + 1, t^2 - s + 1, t^2 : (2t^2 - s + 1) \times (2t^2 + s + 1)\right).$$

Also, by our computer search, we can say what parameters are possible for a triple array with $\lambda_{cc} \leq 10^6$ to be balanced for intersection.

Proposition 2.20 *Let $\mathcal{A}$ be a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with $v = r + c - 1$, $\lambda_{cc} \leq 10^6$ and balanced row-column intersection. Then $\mathcal{A}$ must be one of the following:*

$$TA(10, 3, 3, 2, 3 : 5 \times 6)$$
$$TA(901, 225, 252, 200, 225 : 425 \times 477)$$
$$TA(174725, 43681, 44044, 43320, 43681 : 87001 \times 87725).$$

We conclude this section by mentioning that the definition of the property of balanced for intersection can be extended to more general binary row-column designs and that many such designs possessing the property can be found (cf. [15]). However, our search indicates that triple arrays possessing the property may be very rare indeed; note that in Proposition 2.20 above only the first one of the designs is known to exist.

3 Inner balance with respect to every block

Inner balance for an SBIBD $\mathcal{D}$ is defined with respect to a particular block B_0. In this section we discuss when $\mathcal{D}$ has inner balance with respect to every block. We will need the following notion, introduced by Cameron [7].

Definition 3.1 An SBIBD $\mathcal{D}$ is said to be *quasi*-3 if there exist integers x and y, called triple intersection numbers, such that $|A \cap B \cap C| \in \{x, y\}$ for any three distinct blocks A, B, and C of $\mathcal{D}$.

A design described in Definition 3.1 is often called *quasi-3 for blocks*. Thus because quasi-3 designs are sometimes defined dually, a design is said to be *quasi-3 for points* if the number of blocks incident with sets of three points takes on just two values. The next lemma is straightforward and well-known (cf. [12]).

Lemma 3.2 *A nontrivial symmetric design $\mathcal{D}$ is quasi-3 if and only if every derived design of $\mathcal{D}$ is quasi-symmetric.*

This means that given an SBIBD with inner balance with respect to one particular block, we can formulate a necessary and sufficient condition for the SBIBD to have inner balance with respect to every block.

Theorem 3.3 *Let $\mathcal{D} = (X, \mathcal{B})$ be an SBIBD with inner balance with respect to a block B_0. Then $\mathcal{D}$ has inner balance with respect to every block $B_s \in \mathcal{B}$ if and only if $\mathcal{D}$ is quasi-3.*

Proof Suppose that $\mathcal{D}$ has inner balance with respect to every block $B \in \mathcal{B}$. Then we know by Theorem 2.8 that all the residual designs $\mathcal{D}^B$ are quasi-symmetric with the same intersection numbers, as they must satisfy the same quadratic equation with coefficients in k and λ. By Lemma 2.5 this means that all the derived designs $\mathcal{D}_B$ are quasi-symmetric, and by

Lemma 3.2 we get that $\mathcal{D}$ is quasi-3. Conversely, given that $\mathcal{D}$ is quasi-3 and has inner balance with respect to some block B_0, we know that any residual design $\mathcal{D}^B$ is quasi-symmetric with the same intersection numbers $\mu_{R_\alpha}, \alpha \in \{0, 1\}$ satisfying $\lambda_\star = (k - \lambda)\mu_{R_\alpha} - \mu_{R_\alpha}^2$ of Lemma 2.6, where $\lambda_\star$ is the balance index of the inner design. Hence, every inner design of $\mathcal{D}$ is balanced. □

Remark 3.4 There are SBIBDs which are not quasi-3, but have quasi-symmetric residuals with respect to some blocks. One example is the (78, 22, 6)*-SBIBD given in Tonchev* [18]. *Bracken* [3] *defines* pseudo quasi-3 designs *as SBIBDs with the property that the derived and residual designs with respect to at least one block are quasi-symmetric. Therefore, there might be SBIBDs which have inner balance with respect to one block, but not to another, although no such example is known.*

The quasi-3 designs are not fully classified. A short overview of the general theory of quasi-3 designs and current existence results can be found in [12]. It should be pointed out that some papers cited in this section deal with quasi-3 for blocks and some with quasi-3 for points. In most cases the dual of a quasi-3 design is quasi-3, in which case it is quasi-3 for blocks as well as for points, but this is not always true, which was proved by Bracken and McGuire [2]. However, since we in this paper use only the triple intersection numbers to exclude quasi-3 designs from being inner balanced, the following result shows that for our purposes we do not need to distinguish between the two.

Theorem 3.5 [4] *If the dual design of a quasi-3 design is quasi-3, the two designs must have the same triple intersection numbers.*

We shall now search for potential candidates of SBIBDs with inner balance with respect to every block. We thus go through a list of all known classes first presented for quasi-3 for points in the survey paper by Broughton and McGuire [5]. They divide all known quasi-3 designs into the following five classes.

(C1) SBIBDs with $\lambda \leq 2$.
(C2) Point-hyperplanes in $PG(n, q)$.
(C3) SDP-designs.
(C4) $(4u^2, 2u^2 - u, u^2 - u)$ designs.
(C5) The complementary designs of quasi-3 designs.

We now show that apart from the $(11, 5, 2)$-SBIBD and its complementary design, no quasi-3 design of these five classes has inner balance.

Case $(C1)$ It is known that any (v, k, λ)-SBIBD with $\lambda \leq 2$ is a quasi-3 design with $x = 0$ and $y = 1$ (cf. [12, p. 263]). Together with Theorem 3.3 this tells us that the unique $(11, 5, 2)$-SBIBD has inner balance with respect to every block, and by Corollary 2.11 we already know that it is the only inner balanced design in this class.

Case $(C2)$ In a projective geometry $PG(n, q)$, a point-hyperplane $PG_{n-1}(n, q)$ is an SBIBD which is quasi-3, and a proof for this can be found in [12]. Further, the following result about its residual and derived designs is well-known.

Proposition 3.6 *Let q be a prime power and n a positive integer. If B is a block of the design $\mathcal{D} = PG_{n-1}(n, q)$, then the residual design $\mathcal{D}^B$ is isomorphic to $AG_{n-1}(n, q)$ and, for $n \geq 2$, the derived design $\mathcal{D}_B$ is isomorphic to the q-fold multiple of $PG_{n-2}(n, q)$.* □

Proposition 3.7 *Let $\mathcal{D}$ be a $PG_{n-1}(n, q)$ with $n \geq 2$. Then $\mathcal{D}$ has no inner balance.*

Proof Let $\mathcal{D}$ be a $PG_{n-1}(n, q)$ with parameters (v, k, λ). By Proposition 3.6 we know that the derived design $\mathcal{D}_B$ of $\mathcal{D}$ is isomorphic to the q-fold multiple of $PG_{n-2}(n-1, q)$. Since $\mathcal{D}_B$ has block size λ and q copies of each block we know about the intersection number μ_{D_1} of Lemma 2.5 that $\mu_{D_1} = \lambda$. By Lemma 2.5 this gives that $\mu_{R_0} = 0$, so $\mathcal{D}$ has no inner balance by Corollary 2.7. □

Case $(C3)$ Kantor [13] proved that all non-trivial SDP-designs have the same type of parameters.

Definition 3.8 An SBIBD $\mathcal{D}$ is said to have the *symmetric difference property* (or to be a SDP-design) if the symmetric difference of any three blocks of $\mathcal{D}$ is either a block of $\mathcal{D}$ or the complement of a block of $\mathcal{D}$.

Theorem 3.9 [13]. *Let $\mathcal{D}$ be a nontrivial (v, k, λ) SDP-design with $v \geq 3$. Then there exists a positive integer m such that $v = 2^{2m}, k = 2^{2m-1} \pm 2^{m-1}$, and $\lambda = 2^{2m-2} \pm 2^{m-1}$. Furthermore, the dual design $\mathcal{D}^T$ is an SDP-design.*

A proof of the following result can be found in [12, p. 264].

Proposition 3.10 *Any SDP-design is quasi-3.*

The $(C3)$ class of SDP-designs is a special case of the $(C4)$ class. Therefore we omit the proof of the next proposition, as it will be covered by Proposition 3.13.

Proposition 3.11 *The SDP-designs do not have inner balance.*

Case $(C4)$ There is a conjecture that quasi-3 $(4u^2, 2u^2 - u, u^2 - u)$-SBIBDs exist for all even u, and also that all quasi-3 designs with $\lambda > 2$ and $y < \lambda$ lie in this class. When considering inner balance for this class, the following lemma will be useful.

Lemma 3.12 [4] *If there is a quasi-symmetric $(2u^2 - u, u^2 - u, u^2 - u - 1)$-BIBD with block intersection numbers x and y, where $x < y$. Then $x = \frac{u(u-2)}{2}$ and $y = \frac{u(u-1)}{2}$. In particular, u is even.*

Proposition 3.13 *Suppose that $\mathcal{D}$ is a quasi-3 $(4u^2, 2u^2 - u, u^2 - u)$-SBIBD. Then $\mathcal{D}$ has no inner balance.*

Proof Suppose that $\mathcal{D}$ is a $(4u^2, 2u^2 - u, u^2 - u)$-SBIBD which is quasi-3 with triple intersection numbers x and y. Then x and y are intersection numbers of a derived design with respect to some block, so we can write $x = \mu_{D_0}$ and $y = \mu_{D_1}$. Proposition A.4 gives that the derived design is a $(2u^2 - u, u^2 - u, u^2 - u - 1)$-BIBD, and from Lemma 3.12 we know that $x = \frac{u(u-2)}{2}$ and $y = \frac{u(u-1)}{2}$ for $u > 1$. Further, suppose that $\mathcal{D}$ has inner balance. Then by Lemma 2.5 we can write $3\lambda - k = x + y$. Inserting the given parameters of $\mathcal{D}$ into the left side of this equation gives

$$3\lambda - k = 3(u^2 - u) - (2u^2 - u) = u^2 - 2u,$$

and into the right side

$$x + y = \frac{u(u-2)}{2} + \frac{u(u-1)}{2} = u^2 - \frac{3u}{2}.$$

As the two sides are not equal, this gives us a contradiction and hence we know that $\mathcal{D}$ has no inner balance. □

Case ($C5$) By Proposition 2.4 we know that the inner design of an SBIBD $\mathcal{D}$ is equal to the inner design of $\mathcal{D}'$. As the $(11, 5, 2)$-SBIBD is the only inner balanced design in the previous classes, this leaves us with its complementary design, the $(11, 6, 3)$-SBIBD. This concludes our investigation of the five classes, but some observations remain.

From the Pell equation in Proposition 2.18 we have parameters for a potential family of SBIBDs that are candidates for inner balance. If they exist, we do not know if they are quasi-3, but it is easy to see that they do not belong to any of the five classes in the above list. Recall from the proof of Proposition 2.18 that we consider only $s \geq 26$ and $t \geq 15$. They are not in ($C1$) as

$$\lambda = t^2 - s + 1 = \frac{s^2 - 3s + 2}{3} > 2 \quad \text{for all } s > 4.$$

Nor are they in ($C2$) as $\mu_{R_0} = \frac{t(t-1)}{2} \neq 0$. They are neither in ($C3$) or ($C4$) as $v = 4t^2+2 \not\equiv 0 \pmod 4$. Similar arguments hold for their complementary designs, so they are not in ($C5$).

Finally, the conjecture (cf. Conjecture 47.19 in [8]) that all quasi-3 designs with $\lambda > 2$ and $y < \lambda$ lie within the ($C4$) class together with our results above allow us to conclude this paper by stating the ensuing result for SBIBDs with inner balance with respect to every block.

Conjecture 3.14 *Let $\mathcal{D}$ be an SBIBD which has inner balance with respect to every block. Then $\mathcal{D}$ is the unique* $(11, 5, 2)$*-SBIBD or its complementary design.*

Acknowledgments We would like to thank Frank Wikström for his help with the computer search. Also, we are very grateful for the many valuable suggestions from the reviewers that made this article both stronger and more readable.

Appendix: Preliminaries

Some definitions and results to make the paper readable for non-specialists. Proofs of these preliminary results can be found in [11].

Definition A.1 A *combinatorial design* $\mathcal{D}$ is a pair $(X, \mathcal{B})$, where X is a set of v elements called points, and $\mathcal{B}$ is a collection of b subsets of X called blocks.

(1) If there exist positive constants k, r such that each block contains exactly k points and each point occur in exactly r blocks, then $\mathcal{D}$ is called a *block design*.
(2) A block design is called *complete* if $k = v$ and *incomplete* if $k < v$.
(3) A block design is *balanced* if there exists a positive constant λ such that any 2-subset of X occurs in exactly λ of the blocks. In this case we shall call λ the *index* of the design.

We denote a balanced incomplete block design by BIBD and write the parameters (v, b, r, k, λ) or just (v, k, λ), since b and r then will be obtainable. A BIBD where $v = b$ is said to be *symmetric* and is denoted by SBIBD.

We will need the well-known fundamental identities for BIBDs.

Proposition A.2 *Let $\mathcal{D}$ be a block design with parameters* (v, b, r, k)*, then*

(1) $vr = bk$.
(2) If $\mathcal{D}$ is balanced with index λ, then $\lambda(v - 1) = r(k - 1)$.

If one can talk about a method in this paper, it would be that we are looking at block intersections for subdesigns of SBIBDs.

Definition A.3 Let $\mathcal{D} = (X, \mathcal{B})$ be an SBIBD and let $B_0 \in \mathcal{B}$. The *derived design* of $\mathcal{D}$ with respect to B_0, denoted $\mathcal{D}_{B_0}$, has point set B_0 and the blocks are the sets $B_i \cap B_0$, for $B_i \in \mathcal{B} \setminus \{B_0\}$.

Proposition A.4 *Let $\mathcal{D}$ be a (v, k, λ)-SBIBD and let B be a block of $\mathcal{D}$. Then $\mathcal{D}_B$ is a $(k, v-1, k-1, \lambda, \lambda-1)$-BIBD, provided that $\lambda \geq 2$.*

Definition A.5 Let $\mathcal{D} = (X, \mathcal{B})$ be an SBIBD and let $B_0 \in \mathcal{B}$. The *residual design* of $\mathcal{D}$ with respect to B_0, denoted $\mathcal{D}^{B_0}$, has point set $X \setminus B_0$ and the blocks are the sets $B_i \setminus B_0$, for $B_i \in \mathcal{B} \setminus \{B_0\}$.

Proposition A.6 *Let $\mathcal{D}$ be a (v, k, λ)-SBIBD and let B be a block of $\mathcal{D}$. Then $\mathcal{D}^B$ is a $(v-k, v-1, k, k-\lambda, \lambda)$-BIBD.*

Definition A.7 Let $\mathcal{D} = (X, \mathcal{B})$ be a block design. The *complementary design* of $\mathcal{D}$, denoted $\mathcal{D}'$, has point set X and the blocks are the sets $X \setminus B_i$ for $B_i \in \mathcal{B}$.

Proposition A.8 *Let $\mathcal{D}$ be a (v, b, r, k, λ)-BIBD. Then $\mathcal{D}'$ is a $(v, b, b-r, v-k, b-2r+\lambda)$-BIBD, provided that $b - 2r + \lambda > 0$.*

Definition A.9 Let $\mathcal{D}$ be a combinatorial design. The *dual design* is obtained by interchanging the roles of blocks and points.

Definition A.10 Suppose $\mathcal{D}$ is a block design with blocks $B_0, B_1, \ldots, B_{b-1}$. The distinct cardinalities $|B_i \cap B_j|$, $i \neq j$, are called the *intersection numbers* of $\mathcal{D}$.

Proposition A.11 *A BIBD with $v < b$ has at least two intersection numbers, while an SBIBD has exactly one, equal to λ.*

Definition A.12 A BIBD with exactly two intersection numbers is called a *quasi-symmetric* design.

In addition to combinatorial designs we will consider row-column designs.

Definition A.13 A *row-column design* $\mathcal{A}$ is an $r \times c$ array in which each cell contains exactly one element of some v-set V of symbols.

(1) $\mathcal{A}$ is called *binary* if there is no repetition in any row or column.
(2) $\mathcal{A}$ is called *equireplicate* if every element of V appears the same number of times in $\mathcal{A}$.

References

1. Agrawal H.: Some methods of construction of designs for two-way elimination of heterogeneity. J. Am. Stat. Ass. **61**(316), 1153–1171 (1966).
2. Bracken C., McGuire G.: Duals of quasi-3 designs are not necessarily quasi-3. Des. Codes Cryptogr. **44**(1–3), 157–167 (2007).
3. Bracken C.: Pseudo quasi-3 designs and their applications to coding theory. J. Combin. Des. **17**(5), 411–418 (2009).
4. Broughton W., McGuire G.: Some observations on quasi-3 designs and hadamard matrices. Des. Codes Cryptogr. **18**(1–3), 55–61 (1999).

5. Broughton W., McGuire G.: On the non-existence of quasi-3 designs. Discr. Math. **262**(1–3), 79–87 (2003).
6. Cameron P.J.: Biplanes. Math. Z. **131**, 85–101 (1973).
7. Cameron P.J.: Near-regularity conditions for designs. Geom. Ded. **2**, 213–223 (1973).
8. Colburn C.J., Dinitz J.H.: The CRC Handbook of Combinatorial Designs, 2nd edn. Chapman & Hall/CRC, Boca Raton (2007).
9. Eccleston J., Russell K.: Connectedness and orthogonality in multifactor designs. Biometrika **62**, 341–345 (1975).
10. Goethals J.M., Seidel J.J.: Strongly regular graphs derived from combinatorial designs. Can. J. Math. **22**, 597–614, (1970).
11. M. Hall Jr.: Combinatorial Theory, 2nd edn. Wiley, USA (1998).
12. Ionin Y.J. Shrikhande M.S.: Combinatorics of Symmetric Designs. Cambridge University Press, Cambridge (2006).
13. Kantor W.M.: Symplectic groups, symmetric designs and line ovals. J. Algebra **33**, 43–58 (1975).
14. McSorley J.P., Phillips N.C.K., Wallis W.D., Yucas J.L.: Double arrays, triple arrays and balanced grids. Des. Codes Cryptogr. **35**, 21–45 (2005).
15. Nilson T.: Pseudo-Youden designs balanced for intersection. J. Stat. Plan. Inference **141**, 2030–2034 (2011).
16. Rosen K.H.: Elementary Number Theory, 5th edn. Addison-Wesley, USA (2004).
17. Shrikhande S.S., Bhagwandas S.S.: Duals of incomplete block designs. J. Indian Stat. Assoc. **3**, 30–37 (1965).
18. Tonchev V.D.: Embedding of the Witt-Mathieu system $S(3, 6, 22)$ in a symmetric $2 - (78, 22, 6)$ design. Geom. Dedicata **22**(1), 49–75 (1987).