

Mid-Sweden University. Department of Engineering, Physics and Mathematics
Holmgatan 10, SE-851 70 Sundsvall Phone +46 771 97 50 00

****Row-column designs with adjusted
orthogonality**

Tomas Nilson

Abstract

***Row-column designs with adjusted orthogonality or triple arrays* as they also are called were first studied by statisticians like Agrawal [4]. This because the designs are used in statistical experiments for *two-way elimination of heterogeneity*. Besides presenting theory, construction methods and the main conjecture for these designs we define and study their duals, a special case of *Graeco-latin designs*. A construction of an infinite family of these duals is given.

Acknowledgments

I would like to thank my supervisor Pia Heidtmann for introducing me to design theory in general, for all support under the way, with and between courses and finally for all help with this thesis.

Contents

Introduction	1
1 Preliminaries	3
1.1 Combinatorial designs	3
1.2 Row-column designs	4
2 The designs	7
2.1 Two equivalent definitions	7
2.1.1 Graeco-latin designs	9
2.2 Some concepts for designs	13
2.2.1 Classification of designs	15
3 Results	19
3.1 Triple arrays	19
3.1.1 Triple arrays with $v = r + c - 1$	22
3.1.2 Triple arrays with $v > r + c - 1$	24
3.2 Balanced grids	25
3.2.1 Triple arrays and balanced grids	27
3.3 More results	27
4 Construction methods	31
4.1 Graeco-latin designs	31
4.1.1 The Δ -map	37
4.2 Paley triple arrays	38
4.3 Agrawals method	43
4.3.1 Variations of the method and a generalization	47
5 More about the designs	49
5.1 Isomorphic triple arrays	49
5.2 Existence of SBIBD's	49
5.3 Triple arrays balanced for intersection	50
5.4 A list of small triple arrays	51

A More preliminaries	52
A.1 Preliminaries for 2-factor designs	52
A.1.1 Designs from designs	52
A.1.2 Difference sets	55
A.2 System of distinct representatives	58
Index of notation	59
Bibliography	60

Introduction

Statisticians like to use *balanced incomplete block designs* in experiments, since the analysis then is easy and heterogeneity can be eliminated when choosing the blocks. In this thesis we are going to study ***row-column designs with adjusted orthogonality* which consists of two balanced incomplete block designs, merged so that they are non-interacting and can be analysed with *two-way elimination of heterogeneity*.

We will do the study from a combinatorial point of view, but since the need of these designs arised in statistics we are going to let Shrikhande [40] illustrate such experimental aspects when he writes about Youden squares, the predecessors of our designs.

“Sometimes in a design the position within the block is important as a source of variation, and the experiment gains in efficiency by eliminating the positional effect. The classical example is due to Youden in his studies on tobacco mosaic virus [45] 1937. He found that the response to treatments also depends on the position of the leaf on the plant. If the number of leaves is sufficient so that every treatment can be applied to one leaf of a tree, then we get a ordinary Latin square, in which the trees are columns and the leaves belonging to the same position constitute the rows. But if the number of treatments is larger than the number of leaf positions available, then we must have incomplete columns. Youden used a design in which the columns constituted a balanced incomplete block design, whereas the rows were complete. These designs are known as Youden squares and can be used when two-way elimination of heterogeneity is desired.”

Youden squares are used in all kind of experiments, like in the original agricultural experiment but they also played an important role in World War II.

In the sixties Agrawal [4] amongst others, constructed some ***row-column designs with adjusted ortogonality*. In contrast of Youden squares these are hard to construct. Let us look at the earliest example we have seen. It is an application example by Potthoff [29]. He did not do this experiment for its own sake, but purely to present the design and illustrate the analysis part.

Example 0.1. The experiment is to measure traffic flow at 10 different points around the campus in the mornings. The observations were made every 10 minutes between 8 a.m. and 9 a.m. on 5 mornings in september 1961. A given observation consisted of counting the number of vehicles passing the specified

TOMAS NILSON

point during a 5-minute period. We have three constraints: days, times and locations. A complete experiment would consist of 300 observations but Potthoff used a design with only 30 observations, where each pair of constraints forms well-known structures which can be analysed independently. The raw results of the experiment are given in the following table. For example, 72 vehicles passed location (1) monday between 8 : 00 and 8 : 05.

	8 : 00	8 : 10	8 : 20	8 : 30	8 : 40	8 : 50
<i>Monday</i>	72(1)	101(6)	59(3)	53(4)	10(8)	78(10)
<i>Tuesday</i>	49(2)	50(1)	98(9)	92(10)	38(5)	12(8)
<i>Wednesday</i>	62(3)	13(8)	49(7)	50(2)	73(9)	54(4)
<i>Thursday</i>	52(4)	35(7)	89(1)	82(9)	46(6)	67(5)
<i>Friday</i>	57(5)	55(2)	100(10)	46(6)	34(3)	48(7)

A statistical analysis gives nothing exciting, different locations have different traffic volumes, the traffic builds up to a peak around 8:20 to 8:25 and then declines and the traffic volume is not much different from one week-day to the next. Potthoff points out that the design also utilizes the observers time efficiently since all 30 observations could be made by a single observer, but let us look at the bare design.

1	6	3	4	8	10
2	1	9	10	5	8
3	8	7	2	9	4
4	7	1	9	6	5
5	2	10	6	3	7

This is a **row-column design with adjusted orthogonality. A formal definition will be given in chapter 2 and we will see that both rows and columns as sets constitutes duals of balanced incomplete block designs, merged so that every row-column pair intersects in a constant number of symbols.

Remark 0.2. Potthoff constructed this design strictly by trial and error but he had constructions for a few designs with other parameters, all giving $r \times (r + 1)$ designs. One example is the 11×12 design which he had seen in Beynon [8] 1944 as a duplicate bridge movement. This is the earliest example we have heard of.

Chapter 1

Preliminaries

In this chapter we will make some preliminary definitions on general designs, in order to prepare for the more special definitions in the following chapter. But note that most of the preliminaries can be found in appendix A.

1.1 Combinatorial designs

As we will see, the word design can stand for a lot of structures, but we will take our starting point in a common definition on pairs.

Definition 1.1. *A design \mathcal{D} is a pair $(\mathcal{X}, \mathcal{B})$, where \mathcal{X} is a set of v elements called points, varieties or treatments and \mathcal{B} is a collection of b subsets of \mathcal{X} called blocks.*

1. *If there exist constants k, r such that each block contain k points and each point occur in r blocks, then \mathcal{D} is called a **blockdesign**.*
2. *A design is called **complete** if $k = v$ and **incomplete** if $k < v$.*
3. *A block design is **balanced** if there exists a positive constant λ such that any 2-subset of \mathcal{X} occurs in exactly λ of the blocks. In this case we shall call λ the **index** of the design.*

*We denote a balanced incomplete block design by BIBD and write the parameters (v, b, r, k, λ) or just (v, k, λ) , since b and r then are obtainable. A BIBD where $v = b$ is said to be **symmetric** and is denoted by SBIBD.*

A design is often given by a list of its blocks but the incidence structure can also be represented by a matrix.

Definition 1.2. *An **incidence matrix** of a design \mathcal{D} is a $(0, 1)$ -matrix whose rows are indexed by the points of \mathcal{D} , columns are indexed by the blocks of \mathcal{D} , and the (x, B) -entry is equal to 1 if and only if $x \in B$.*

These matrices are useful tools for us since they open for calculations and results from linear algebra. Here we just state a standard result which gives a matrix definition of a BIBD. Let I_n denote the $n \times n$ identity matrix and let $J_{m,n}$ denote the $m \times n$ matrix in which every entry is equal to 1. If J is square of order n , we will write J_n .

Theorem 1.3. *A $(0,1)$ -matrix A with v rows and b columns, is the incidence matrix of a (v, k, λ) BIBD if, and only if, the following conditions are satisfied.*

1. $J_v A = k J_{v,b}$, where $2 \leq k < v$;
2. $AA^T = (r - \lambda)I_v + \lambda J_v$.

A proof can be found in chapter 1.2 in Whitehead [44].

A design is an incidence structure given by ordered pairs (x, B) , but sometimes we want to look at this structure the other way round.

Definition 1.4. *Let \mathcal{D} be a design. The **dual design** \mathcal{D}' of \mathcal{D} is obtained by interchanging the roles of blocks and points.*

Observation 1.5. Let \mathcal{D} be a design with incidence matrix A . Then A^T is the incidence matrix for the dual design \mathcal{D}' .

1.2 Row-column designs

A block design as in definition 1.1 is an incidence structure of ordered pairs, points and blocks. These are called the **factors** or **constraints** of the design. Now we are going to look at 3-factor designs and they are often represented as row-column designs, where the factors are rows, columns and symbols. Note that 3-factor designs do not satisfy definition 1.1 of designs. Since we will use the common parameters for both 2- and 3-factor design, parameters can have different meaning in the two cases.

Definition 1.6. *A row-column design \mathcal{A} is a rectangular $r \times c$ array, where each cell contains exactly one element of some v -set V of symbols.*

1. \mathcal{A} is called **binary** if there is no repetition in any row or column.
2. \mathcal{A} is called **equireplicate** if every element of V appears the same number k of times in \mathcal{A} .

In this thesis we are looking at designs that are both binary and equireplicate. Our prime notation for such a design is $RCD(v, k : r \times c)$, where k is the replication number. In some cases we use other acronyms, and add more parameters. We shall often consider the set of all elements of a row. Some authors refer to

CHAPTER 1. PRELIMINARIES

this set as the support of the row; and similarly for columns, but we will only write row and it will be clear from the context that we refer to a set.

The best known row-column design are probably the latin squares, the design with $r = c = v = k$.

Definition 1.7. *A latin square is an arrangement of n symbols in a $n \times n$ array, such that every symbol occurs once in each row and in each column.*

Like Shrikhande [40] told us in the introduction, there is a need for designs where the parameters differ. The following definition will cover such row-column designs.

Definition 1.8. *An $m \times n$ rectangle formed by v symbols, $m, n \leq v$, both m and n are not equal to v is called a **latin rectangle** if every symbol occurs at most once in each row and in each column.*

The statistical analysis of a general latin rectangle will normally be complicated. However, it will be a lot easier if we let the rows and columns form well-known structures.

Definition 1.9. *A **Youden-square** is an arrangement of v symbols in k rows and v columns such that*

1. *every symbol occurs exactly once in each row;*
2. *the columns form a (v, k, λ) SBIBD.*

We construct Youden squares by writing the blocks of a SBIBD as columns in a row-column design. Then rearrange the elements within the columns so that every element occurs exactly once in each row. The next theorem by Levi [20] tells us that it is always possible to do the rearrangement within blocks, but we will quote the proof of Raghavarao [36] using system of distinct representatives (SDR) from definition A.35, since it will be of further interest.

Theorem 1.10. *A Youden square can always be constructed from a SBIBD.*

Proof. Let \mathcal{D} be a (v, k, λ) SBIBD. Theorem A.1 gives that the replication number r of \mathcal{D} is equal to k . Write the blocks of \mathcal{D} as columns. Then any h columns, $1 \leq h \leq v$ contain between themselves hr symbols of which at least h are distinct, since each symbol can occur at most r times in these h columns. Thus, from Halls theorem A.37, an SDR exists for the v columns, and this SDR will be a permutation of the v symbols of \mathcal{D} . Bring this SDR to the first row. Deleting the first row we find that each column now contains $r - 1$ symbols, and h of these columns will contain $h(r - 1)$ symbols of which at least h are distinct. Hence another SDR exists for the columns, and we bring this SDR to the second row. Continuing similarly, we can prove that the k rows can be so arranged that every symbol occurs exactly once in each of the rows. \square

TOMAS NILSON

Let us construct a Youden square using the proof of theorem 1.10 by Raghavarao.

Example 1.11. Here is a $(7, 3, 1)$ SBIBD where the blocks has been written as columns.

$$\begin{array}{cccccc} 0 & 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 3 & 5 & 3 & 4 & 3 & 4 \\ 2 & 4 & 6 & 5 & 6 & 6 & 5 \end{array}$$

We choose a SDR for the columns which becomes the first row in our row-column design.

$$0 \ 3 \ 5 \ 1 \ 4 \ 6 \ 2$$

The remainings of the SBIBD are

$$\begin{array}{cccccc} 1 & 0 & 0 & 3 & 1 & 2 & 4 \\ 2 & 4 & 6 & 5 & 6 & 3 & 5 \end{array}$$

and we choose a second SDR which becomes the second row.

$$\begin{array}{cccccc} 0 & 3 & 5 & 1 & 4 & 6 & 2 \\ 1 & 4 & 0 & 3 & 6 & 2 & 5 \end{array}$$

The SBIBD has now just one element left in each block and these form the last SDR, which becomes the third row in our $RCD(7, 3 : 3 \times 7)$ which is a Youden square.

$$\begin{array}{cccccc} 0 & 3 & 5 & 1 & 4 & 6 & 2 \\ 1 & 4 & 0 & 3 & 6 & 2 & 5 \\ 2 & 0 & 6 & 5 & 1 & 3 & 4 \end{array}$$

Some SBIBD's can be written directly as a Youden square, since the rearranging follows by the construction.

Example 1.12. A $(7, 3, 1)$ SBIBD constructed by theorem A.17 on the set $\{1, 2, 4\}$ in $GF(7)$ written as a Youden square.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 4 & 5 & 6 & 0 & 1 & 2 & 3 \end{array}$$

Chapter 2

The designs

The intension of this chapter is to define the designs we have set out to study and investigate the properties of their duals. We will also compare them with other designs and consider how to classify a given design.

2.1 Two equivalent definitions

From the introduction we know that ****row-column** designs with adjusted orthogonality can be seen as descendants of Youden squares. But now we want both rows and columns to be incomplete. This means that the number of symbols is greater than the number of rows and the number of columns. Fisher's inequality A.7 then tell us that neither the rows nor the columns can constitute BIBD's as blocks, but they can be two duals of BIBD's.

Definition 2.1. *Let \mathcal{D} be an $r \times c$ row-column design on v symbols that satisfies*

- 1. the rows are the dual of a BIBD,*
- 2. the columns are the dual of a BIBD,*
- 3. every row intersects every column in a constant number of symbols,*

*then \mathcal{D} is called a ****row-column design with adjusted orthogonality.***

Remark 2.2. We will discuss adjusted orthogonality later in this chapter but for a row-column design it is equivalent with property (3). The two stars stands for that both rows and columns are duals of BIBD's. We use them since we here are looking at a subclass of row-column designs with adjusted orthogonality.

Definition 2.1 is about the underlying structure of the design, but recently McSorley et al [24] introduced another definition which tell us about the properties of the row-column design as an array. Let us formulate it immediately since we will use both definitions.

Definition 2.3. Let \mathcal{D} be a binary $r \times c$ row-column design on v symbols, equireplicate with replication number k , where $k < r, c$. If \mathcal{D} satisfies

A1: any two distinct rows intersects in λ_{rr} symbols;

A2: any two distinct columns intersects in λ_{cc} symbols,

then \mathcal{D} is called a **double array**, denoted by $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$.
If \mathcal{D} further satisfies

A3: any row and column intersects in λ_{rc} symbols,

then \mathcal{D} will be called a **triple array**, denoted by $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$.

Example 2.4. A $TA(14, 4, 4, 3, 4 : 7 \times 8)$ constructed by Agrawal [4] where we have labeled the rows and columns. The dual of the rows has 14 blocks, one for each symbol in the triple array. To find them: look at all symbols in the first row, they stand for the eight blocks in the dual which contain the point 1. One of these blocks are $B_9 = \{1, 2, 5, 7\}$.

	1	2	3	4	5	6	7	8
1	2	12	3	5	10	9	1	8
2	3	6	13	9	2	11	4	10
3	5	4	7	10	11	3	12	14
4	1	8	5	4	6	12	13	11
5	13	14	9	6	5	7	2	12
6	14	3	8	1	7	6	10	13
7	11	2	4	14	8	1	7	9

Remark 2.5. Note that we, unlike most authors on triple arrays do not allow $k = r$ or $k = c$ in the definition. This is to avoid trivial cases. If $k = r = c$, then the triple array would be equivalent to a latin square since every symbol occurs exactly once in every row and in every column. If $k = r < c$, then it would be equivalent to a Youden square. To see this, first use double counting on the total number of symbols in a triple array, this gives the equation $vk = rc$, and $k = r$ implies $v = c$. Hence we know that every symbol here occurs exactly once in every row. Corollary 2.7 from the end of this subsection then gives that the columns which are incomplete, are the dual of a BIBD. Since the number of points (columnindices of the TA) is equal to the number of blocks (symbols of the TA), the dual of the columns is a SBIBD by definition and theorem A.8 then tells us that the columns also form a SBIBD, so definition 1.9 of Youden squares is satisfied.

Sometimes it is convenient to look at a **row-column design with adjusted orthogonality since it tell us about the structure, and we also know a lot about how to construct BIBD's. Another time we want to look at details of the array,

CHAPTER 2. THE DESIGNS

like the substructure double array or do a computersearch, and then it is better to look at the triple array. But we need to know that these two definitions are equivalent.

Theorem 2.6. *Triple arrays and **row-column designs with adjusted orthogonality are equivalent.*

Proof. Let \mathcal{D} be a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$. First we will show that \mathcal{D} satisfies definition 2.1.

(i) The rows as sets of symbols satisfy proposition A.6 and are a dual to a BIBD, since each row contains c symbols, each symbol occurs in k rows, $k < v$, and each pair of distinct rows intersects in λ_{rr} symbols. (ii) The columns as sets of symbols satisfy proposition A.6, since each column contains r symbols, each symbol occurs in k columns, $k < v$, and each pair of distinct columns intersects in λ_{cc} symbols. (iii, A3) That any pair of row and column intersects in a constant number of symbols is given directly in both definitions.

Next we will show that a **row-column design with adjusted orthogonality satisfies definition 2.3. Since the dual of a BIBD is a design, the rows and columns are sets and there is no repetition. From proposition A.6 we know that each symbol is incident with a constant number of rows, so the array is equireplicate. (A1) Condition 4 in proposition A.6 tells us that each pair of rows intersects in a constant number of symbols. (A2) The corresponding arguments also holds for the columns. □

We will work with two definitions but will only use one notation for parameters, the general notation for row-column designs which for a triple array is $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$.

From the proof of theorem 2.6 we know the parameters for the duals of rows and columns.

Corollary 2.7. *Suppose D is a $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$. Then*

1. *the dual of the rows is a BIBD with parameters $(r, v, c, k, \lambda_{rr})$.*
2. *the dual of the columns is a BIBD with parameters $(c, v, r, k, \lambda_{cc})$.*

Remark 2.8. Corollary 2.7 tells us that the rows in example 2.4 are the dual of a $(7, 14, 8, 4, 4)$ BIBD, and the columns are the dual of $(8, 14, 7, 4, 3)$ BIBD.

2.1.1 Graeco-latin designs

What is a dual of a row-column design? For 2-factor designs like BIBD's, definition 1.4 tells us that we just interchanges the roles of points and blocks, and there are only these two possible permutations of the factors. For 3-factor designs there are six possible permutations and even more ways to represent them, but we will

keep the idea of interchanging the roles of points and positions. So in a dual of a row-column design we let the symbols denote blocks and take row-column pairs as points. Eccleston and Street [17] wrote as follows.

Definition 2.9. *We will define the dual of a RCD to be the block design with two sets of treatments in the following way. The rows of the RCD will correspond to the first set of treatments, the columns will correspond to the second sets of treatments and the treatments will correspond to the blocks of the dual design.*

Let us rewrite the definition by Eccleston and Street in a more explicit way.

Definition 2.10. *A design $\mathcal{D} = ((\mathcal{X} \times \mathcal{Y}), \mathcal{B})$ is a **Graeco-latin design** if $(\mathcal{X}, \mathcal{B})$ and $(\mathcal{Y}, \mathcal{B})$ are designs and every ordered pair (x_i, y_j) , $i = 1, 2, \dots, |\mathcal{X}|$, $j = 1, 2, \dots, |\mathcal{Y}|$ of $(\mathcal{X} \times \mathcal{Y})$ occurs exactly once in the design. Moreover,*

1. *if \mathcal{D} has constant block size, then \mathcal{D} is said to be a **Graeco-latin block design**,*
2. *if there exists a constant λ such that any given $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ occur together in λ blocks, not necessarily in the same ordered pair, then \mathcal{D} is said to be **balanced**,*
3. *if there exists a constant μ such that for each pair (s_1, s_2)*

$$|\{x : x \in (B_{s_1} \cap B_{s_2})\}| + |\{y : y \in (B_{s_1} \cap B_{s_2})\}| = \mu,$$

*then \mathcal{D} is said to be **linked**.*

Remark 2.11. Note that there are non-equivalent definitions of Graeco-latin designs in the literature. Our definition is more restrictive than some authors, since we require that every ordered pair in $\mathcal{X} \times \mathcal{Y}$ shall occur exactly once. This means that we are only looking at duals of row-column designs. Preece [32] and Seberry [39] have this orthogonality between the two sets of symbols as an additional property. McSorley [25] calls two BIBD's that can form a Graeco-latin block design for *matching BIBD's*.

Remark 2.12. One could describe the Graeco-latin design as two superimposed designs, like in Graeco-latin squares. This explains the name.

Let us look at a small and somewhat trivial example.

Example 2.13. A Graeco-latin block design $\mathcal{D} = ((\mathcal{X} \times \mathcal{Y}), \mathcal{B})$, which is balanced and linked with $\lambda = 2$ and $\mu = 3$.

$$\begin{aligned} B_0 &= \{(0, 0), (1, 1)\} \\ B_1 &= \{(0, 1), (1, 2)\} \\ B_2 &= \{(0, 2), (1, 0)\} \end{aligned}$$

CHAPTER 2. THE DESIGNS

Note that $(\mathcal{X}, \mathcal{B})$ here is a complete block design and $(\mathcal{Y}, \mathcal{B})$ is a SBIBD. We can also represent \mathcal{D} by the dual, a row-column design,

$$\begin{array}{ccc} 0 & 1 & 2 \\ 2 & 0 & 1 \end{array}$$

which is a Youden square.

We would like to know what conditions a Graeco-latin design has to satisfy, in order to be a dual of a double or triple array.

Theorem 2.14. *Let $\mathcal{D} = ((\mathcal{X} \times \mathcal{Y}), \mathcal{B})$ be a Graeco-latin design. Then \mathcal{D} is a dual of a double array if and only if both $(\mathcal{X}, \mathcal{B})$ and $(\mathcal{Y}, \mathcal{B})$ are BIBD's.*

Proof. Let $\mathcal{D} = ((\mathcal{X} \times \mathcal{Y}), \mathcal{B})$ be a Graeco-latin design where both $(\mathcal{X}, \mathcal{B})$ and $(\mathcal{Y}, \mathcal{B})$ are BIBD's. From \mathcal{D} we will construct a $|\mathcal{X}| \times |\mathcal{Y}|$ row-column design \mathcal{A} on $|\mathcal{B}|$ symbols satisfying the necessary parts of definition 2.3 in order to be a double array. The rows and columns are labeled with the elements of \mathcal{X} and \mathcal{Y} respectively and we take the block indices s as symbols. This agrees with definition 2.9 of duals of row-column designs, so \mathcal{A} will be the dual of \mathcal{D} . Each pair (x, y) occurs exactly once in \mathcal{D} by definition 2.10, so we have unique ordered triplets (x, y, s) , and takes $\mathcal{A}(x, y) = s$. That $(\mathcal{X}, \mathcal{B})$ and $(\mathcal{Y}, \mathcal{B})$ are block designs gives that \mathcal{A} is binary since an x or a y occurs not more than once in a block in \mathcal{D} . It also tells us that the block size is constant, which gives that \mathcal{A} is equireplicate. In \mathcal{D} , $(\mathcal{X}, \mathcal{B})$ is a BIBD. In \mathcal{A} we have reversed roles and regard the rows as sets of symbols s , so the rows will be a dual of a BIBD. The same argument holds for $(\mathcal{Y}, \mathcal{B})$ and columns. So \mathcal{A} is a double array by definition 2.3.

From an $|\mathcal{X}| \times |\mathcal{Y}|$ double array \mathcal{A} on $|\mathcal{B}|$ symbols we construct a design \mathcal{D} with pointset $\mathcal{X} \times \mathcal{Y}$, so that (x, y) is in a block B_s of \mathcal{D} if $\mathcal{A}(x, y) = s$. We will show that this dual arrangement is a Graeco-latin design. Since the rows and columns of \mathcal{A} are duals of BIBD's respectively by theorem 2.6, we know that $(\mathcal{X}, \mathcal{B})$ and $(\mathcal{Y}, \mathcal{B})$ are BIBD's. Every cell in the double array is occupied by one symbol, so every pair in $\mathcal{X} \times \mathcal{Y}$ will occur in \mathcal{D} exactly once. So \mathcal{D} satisfies the required part of definition 2.10. \square

Theorem 2.15. *Let \mathcal{D} be a Graeco-latin design where both $(\mathcal{X}, \mathcal{B})$ and $(\mathcal{Y}, \mathcal{B})$ are BIBD's. Then \mathcal{D} is a dual of a triple array if, and only if, \mathcal{D} is balanced. Moreover, \mathcal{D} is a dual of a triple array with $v = r + c - 1$ if, and only if, \mathcal{D} is linked and balanced, or equivalent, \mathcal{D} is linked and $|\mathcal{B}| = |\mathcal{X}| + |\mathcal{Y}| - 1$.*

Proof. We will reuse the constructions \mathcal{A} and \mathcal{D} from the proof of theorem 2.14. The dual statement of balanced, that is condition (2) in definition 2.10, is that any row x and column y in \mathcal{A} will intersect in λ symbols. So \mathcal{A} is a triple array by definition 2.3.

The dual statement of linked, that is condition (3), is that any pair of symbols s_1, s_2 will meet in a total of μ rows and columns. This defines an array we will

meet in definition 3.10, a *balanced grid*. We will use it here in order to get the Graeco-latin design results complete. So \mathcal{A} is a balanced grid and since \mathcal{A} also is a triple array, theorem 3.14 gives that \mathcal{A} satisfies $v = r + c - 1$.

If \mathcal{D} is balanced, its dual \mathcal{A} is a triple array and if \mathcal{D} is linked, then \mathcal{A} is a balanced grid. Theorem 3.14 tells us that an array \mathcal{A} can be both a triple array and a balanced grid, if and only if $v = r + c - 1$.

Identifying the roles of \mathcal{B} , \mathcal{X} and \mathcal{Y} , and taking $|\mathcal{B}| = |\mathcal{X}| + |\mathcal{Y}| - 1$ gives that $v = r + c - 1$ for the dual \mathcal{A} . If \mathcal{D} is linked, then \mathcal{A} is a balanced grid. So \mathcal{A} is a triple array by theorem 3.14. \square

Remark 2.16. We would like to write just linked without parameters in the last condition of theorem 2.15, but it is an open problem whether there exist double arrays with $v < r + c - 1$. See McSorley et al [24].

Let us look at a non-trivial example.

Example 2.17. A Graeco-latin blockdesign \mathcal{D} where $(\mathcal{X}, \mathcal{B})$ is a $(4, 3, 6)$ BIBD, and $(\mathcal{Y}, \mathcal{B})$ is a $(9, 3, 1)$ BIBD.

$$\begin{array}{l|l} B_0 = \{(1, 0), (2, 1), (3, 2)\} & B_6 = \{(3, 0), (1, 5), (0, 7)\} \\ B_1 = \{(1, 3), (2, 4), (3, 5)\} & B_7 = \{(0, 1), (3, 3), (1, 8)\} \\ B_2 = \{(1, 6), (3, 7), (2, 8)\} & B_8 = \{(1, 2), (3, 4), (0, 6)\} \\ B_3 = \{(0, 0), (2, 3), (3, 6)\} & B_9 = \{(2, 0), (1, 4), (0, 8)\} \\ B_4 = \{(3, 1), (0, 4), (2, 7)\} & B_{10} = \{(1, 1), (0, 5), (2, 6)\} \\ B_5 = \{(0, 2), (2, 5), (3, 8)\} & B_{11} = \{(2, 2), (0, 3), (1, 7)\} \end{array}$$

Further \mathcal{D} is balanced and linked with $\lambda = \mu = 3$, so \mathcal{D} is a dual of a triple array \mathcal{A} by theorem 2.15. It is a $TA(12, 3, 3, 1, 3, : 4 \times 9)$ and we construct it by labeling the rows with elements from \mathcal{X} , the columns with elements from \mathcal{Y} and taking $\mathcal{A}(i, j) = s$ if $(x_i, y_j) \in B_s$ in \mathcal{D} .

$$\mathcal{A} = \begin{array}{cccccccc} 3 & 7 & 5 & 11 & 4 & 10 & 8 & 6 & 9 \\ 0 & 10 & 8 & 1 & 9 & 6 & 2 & 11 & 7 \\ 9 & 0 & 11 & 3 & 1 & 5 & 10 & 4 & 2 \\ 6 & 4 & 0 & 7 & 8 & 1 & 3 & 2 & 5 \end{array}$$

We will constuct Graeco-latin designs in chapter 4.

Notes 2.18. Several early authors like Preece [32], Sterling and Wormald [42] and Seberry [39] have studied and used Graeco-latin designs and their connection to row-column designs. However, the terminology and thereby the theory is unclear even if there were attempts like Preece in [32] and [33] to straighten the terminology out. Further, we have not seen explicit properties being stated for a Graeco-latin design in order to be a dual of a triple array as in theorem 2.15.

2.2 Some concepts for designs

Now when we have definitions of the designs we have set out to study and some of their relatives we will look at them again in the light of some concepts and see what differs. One purpose of this is to learn how to classify a given design.

We defined *factors* or *constraints* in the beginning of section 1.2. Suppose that we are considering two 3–constraint designs. They can look very different and nevertheless be two representations of the same structure. We need tools to classify designs, which goes beyond the properties of representations. Preece [32] tried to do this leaning on Pearce [26] among others. The main idea is to look at constraint-pair relations, which can be regarded as projections of the design. We will give a short introduction to this approach, only considering 3-constraint designs.

A *constraint*, like the column-constraint in a row-column design consists of *levels*, that is the actual columns. If every level in a constraint has the same number of elements (units, cells), we say that the design is *proper with respect to* that constraint. So a equireplicate row-column design is proper with respect to the symbols. If the design is proper to all of its constraints, we say that it is *fully proper*.

Suppose that we are looking at the constraints p and q in a design, with k_p and k_q levels respectively. Then the relationship between these two constraints can be specified by the $k_p \times k_q$ incidence matrix N_{pq} , whose (i, j) element is the number of times the i th level of p occurs in conjunction with the j th level of q . If the design is proper with respect to these constraints, the relationship can be summed up by the matrix $N_{pq}N_{pq}^T$.

Example 2.19. We know from theorem 1.3 that if we take constraints p as the points, and q as the blocks of a BIBD with parameters (v, b, r, k, λ) , then

$$N_{pq}N_{pq}^T = (r - \lambda)I + \lambda J.$$

The relationship in example 2.19 differs from the relation between rows and columns in a row-column design, where every row is incident with every column. Let us define these two important types of relationships.

Definition 2.20. *If a design is proper with respect to its constraints p and q , we shall say that constraint p is **orthogonal** with respect to q if*

$$N_{pq}N_{pq}^T = cJ$$

for some c and **totally balanced** with respect to q if

$$N_{pq}N_{pq}^T = xI + yJ$$

for some x and y .

We will say that the relationship of p with respect to q is of type O if they are orthogonal, or of type T if there is total balance.

Now we want to compile the pair-relations. There are six possible relations of a 3-constraint design, but here we only consider three of them. Note that if $N_{rc}N_{rc}^T = cJ_r$ then $N_{cr}N_{cr}^T = rJ_c$ so the type O is preserved when we swap two constraints. This is not the case if we swap the constraints in example 2.19. Then we are looking at the dual of a BIBD and this is not of type T. We know from theorem A.8 that the dual of a BIBD D is a BIBD if, and only if, D is symmetric.

We will say that a design with tree constraints is of type X:YZ if the relationship between the second constraint with respect to the first is of type X, that of the third constraint with respect to the first is of type Y, and that of the third constraint with respect to the second is of type Z. To make this easier to remember we can write $2_1 : 3_13_2$, but note that the ordering of constraints is arbitrary. A design is said to be *non-orthogonal* if any of the relationships X, Y and Z is not of type O.

Later we will see that both double and triple arrays are of type $T : TO$, and therefore Preece sometimes extends the notation to $X : YZ(Q_1, Q_2, Q_3)$, where Q_i will denote the *overall* relationship of the i th constraint with respect to the rest of the design. This overall relationship in Preece [32] is a bit vague but if we have a fully proper designs with relations T:TO, and $N_{21}N_{31}^T = kJ$, then Preece says that both second and third constraints have *overall total balance* with respect to the first. He calls such a design a fully proper $T : TO(Q, T, T)$ design, and it has the structure of a triple array.

Instead of overall total balance we will say that the two constraints have adjusted orthogonality with respect to the first. This is a more common way to put it and the property was defined by Eccleston and Russel [15]. One way to express it which explains the name is: “Given factors A, B and C . Adjust A and B for C . If the results are orthogonal to each other then A and B are defined to have adjusted orthogonality with respect to C ”, but we will use their more practical definition.

Definition 2.21. *We will say that an equi-replicate RCD($v, k : r \times c$) is **adjusted-orthogonal** if and only if*

$$N_{rs}N_{cs}^T = kJ_{r,c}$$

where r, c and s denote the constraints rows, columns and symbols respectively.

There are at least tree equivalent statistical definitions as well, but in a row-column design it is clear what adjusted orthogonality means.

Observation 2.22. Let D be a row-column design and let r, c and s denote the constraints rows, columns and symbols respectively. Then

$$N_{rs}N_{cs}^T = kJ_{r,c}$$

tells us that each pair of row and column intersects in k symbols.

CHAPTER 2. THE DESIGNS

Adjusted orthogonality is clearly important for statisticians. For example, Eccleston and John [22] wrote. “*If a row-column design has the property of adjusted orthogonality then one need consider the component designs only*”.

Remark 2.23. From our perspective, a design first reach to be a double array and then we check for adjusted orthogonality. However, there are lots of row-column designs with adjusted orthogonality which do not posses the properties of a double array.

Let us write a definition for triple arrays using products of incidence matrices. Note that already Potthoff [29] wrote such a definition, but it was in the special case for $r \times (r + 1)$ arrays.

Definition 2.24. *Let \mathcal{D} be a binary $r \times c$ row-column design on v symbols, equireplicate with replication number k , where $k < r, c$, and let us order the constraints symbols, rows and columns by 1, 2 and 3. If \mathcal{D} satisfies*

$$(1) N_{21}N_{21}^T = aI_r + (c - a)J_r,$$

$$(2) N_{31}N_{31}^T = bI_c + (r - b)J_c,$$

for some $a, b > 0$, then \mathcal{D} is a double array. If \mathcal{D} further satisfies

$$(3) N_{21}N_{31}^T = kJ_{r,c},$$

then \mathcal{D} is a triple array.

2.2.1 Classification of designs

Let us classify some row-column designs by the matrixproducts in definitions 2.20 and 2.21. Let s, r and c denote the constraints symbols, rows and columns respectively.

Example 2.25. Consider the Latin square L .

$$L = \begin{array}{c|cccc} & A & B & C & D \\ \hline a & 0 & 1 & 2 & 3 \\ b & 1 & 0 & 3 & 2 \\ c & 2 & 3 & 0 & 1 \\ d & 3 & 2 & 1 & 0 \end{array}$$

We have incidence matrices $N_{rs} = N_{cs} = N_{cr} = J_4$, and get the products

$$N_{rs}N_{rs}^T = 4J_4$$

$$N_{cs}N_{cs}^T = 4J_4$$

$$N_{cr}N_{cr}^T = 4J_4$$

TOMAS NILSON

Hence L is a $O : OO$ design, and since all relations are orthogonal, the design is called orthogonal. Further we have

$$N_{rs}N_{cs}^T = 4J_4.$$

So L is adjusted orthogonal. Note that in a Latin square, all pairs of factors have the same relation.

Example 2.26. Let us classify a Youden square Y .

$$Y = \begin{array}{c|cccc} & A & B & C & D \\ \hline a & 0 & 3 & 2 & 1 \\ b & 1 & 0 & 3 & 2 \\ c & 2 & 1 & 0 & 3 \end{array}$$

We have two trivial incidence matrices $N_{rs} = N_{rc} = J_{3,4}$, and the third is

$$N_{cs} = \begin{bmatrix} 1 & 1 & 1 & - \\ 1 & 1 & - & 1 \\ 1 & - & 1 & 1 \\ - & 1 & 1 & 1 \end{bmatrix}$$

So the products are

$$N_{rs}N_{rs}^T = 4J_3$$

$$N_{cs}N_{cs}^T = I_4 + 2J_4$$

$$N_{cr}N_{cr}^T = 3J_4.$$

Hence Y is a $O : TO$ design, an example on an non-orthogonal design. The last product

$$N_{rs}N_{cs}^T = 3J_{3,4}$$

tells us that Y is adjusted orthogonal.

Example 2.27. Let us look at a double array D .

$$D = \begin{array}{c|cccc} & A & B & C & D \\ \hline a & 0 & 1 & 2 & 3 \\ b & 2 & 0 & 4 & 5 \\ c & 5 & 3 & 1 & 4 \end{array}$$

The incidence matrices are

$$N_{rs} = \begin{bmatrix} 1 & 1 & 1 & 1 & - & - \\ 1 & - & 1 & - & 1 & 1 \\ - & 1 & - & 1 & 1 & 1 \end{bmatrix}$$

CHAPTER 2. THE DESIGNS

$$N_{cs} = \begin{bmatrix} 1 & - & 1 & - & - & 1 \\ 1 & 1 & - & 1 & - & - \\ - & 1 & 1 & - & 1 & - \\ - & - & - & 1 & 1 & 1 \end{bmatrix}$$

and $N_{cr} = J_{4,3}$. We get products

$$N_{rs}N_{rs}^T = 2I_3 + 2J_3$$

$$N_{cs}N_{cs}^T = 2I_4 + J_4$$

$$N_{cr}N_{cr}^T = 3J_4.$$

So D is a $T : TO$ design. The last product is

$$N_{rs}N_{cs}^T = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 3 & 1 & 2 & 2 \\ 1 & 2 & 2 & 3 \end{bmatrix} \neq kJ.$$

Hence D is not adjusted orthogonal.

So far we have looked at well-known examples, but in statistical papers we meet a lot of different constructions and representations of designs. Such a design can have clearly defined properties within the representation. But if we want to determine and compare the structure with other designs we can look at the factor-pair projections and then try to classify.

Example 2.28. Here is a design from Sterling and Wormald [42].

<i>Ab</i>	<i>Dc</i>	<i>Gb</i>	<i>Aa</i>	<i>Ba</i>	<i>Ca</i>	<i>Ad</i>	<i>Bd</i>	<i>Cb</i>	<i>Ac</i>	<i>Bb</i>	<i>Cc</i>
<i>Bc</i>	<i>Ed</i>	<i>Hc</i>	<i>Dd</i>	<i>Ec</i>	<i>Fd</i>	<i>Fa</i>	<i>Da</i>	<i>Ea</i>	<i>Eb</i>	<i>Fc</i>	<i>Db</i>
<i>Cd</i>	<i>Fb</i>	<i>Id</i>	<i>Gc</i>	<i>Hd</i>	<i>Ic</i>	<i>Hb</i>	<i>Ib</i>	<i>Gd</i>	<i>Ia</i>	<i>Ga</i>	<i>Ha</i>

It does not satisfy our definition 1.6 for row-column designs precisely, but it is close and we can immediately see four possible constraints, the rows, columns and two kinds of letters, capitals and small. We believe that Sterling and Wormald constructs binary designs, and since letters occurs several times in a row we exclude the row-constraint and check the relations amongst pairs of the other tree. Let L, l and c denote the constraints capital letters, small letters and columns respectively. We label the columns $s = 1, 2, \dots, 12$. The incidence matrices are

$$N_{lL} = \begin{array}{cccccccccc} & A & B & C & D & E & F & G & H & I \\ a & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ b & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ c & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ d & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

TOMAS NILSON

		1	2	3	4	5	6	7	8	9	10	11	12
	<i>A</i>	1	–	–	1	–	–	1	–	–	1	–	–
	<i>B</i>	1	–	–	–	1	–	–	1	–	–	1	–
	<i>C</i>	1	–	–	–	–	1	–	–	1	–	–	1
	<i>D</i>	–	1	–	1	–	–	–	1	–	–	–	1
$N_{Lc} =$	<i>E</i>	–	1	–	–	1	–	–	–	1	1	–	–
	<i>F</i>	–	1	–	–	–	1	1	–	–	–	1	–
	<i>G</i>	–	–	1	1	–	–	–	–	1	–	1	–
	<i>H</i>	–	–	1	–	1	–	1	–	–	–	–	1
	<i>I</i>	–	–	1	–	–	1	–	1	–	1	–	–
		1	2	3	4	5	6	7	8	9	10	11	12
	<i>a</i>	–	–	–	1	1	1	1	1	1	1	1	1
$N_{lc} =$	<i>b</i>	1	1	1	–	–	–	1	1	1	1	1	1
	<i>c</i>	1	1	1	1	1	1	–	–	–	1	1	1
	<i>d</i>	1	1	1	1	1	1	1	1	1	–	–	–

The first matrix product

$$N_{iL}N_{iL}^T = 9I_4$$

tells us that the two sets of letters are orthogonal. Each pair meet exactly once.

$$N_{Lc}N_{Lc}^T = 3I_9 + J_9$$

$$N_{lc}N_{lc}^T = 3I_4 + 6J_4$$

These product indicates total balance and since the block sizes are constant, theorem 1.3 tells us that both N_{Lc} and N_{lc} are indeed incidence matrices of BIBD's. The last product

$$N_{Lc}N_{lc}T = 3J_4$$

tells us that the two sets of letters are adjusted orthogonal with respect to the columns. So, it is a fully proper $T : TO(Q, T, T)$ design, a balanced Graeco-latin blockdesign with columns as blocks, containing pairs of letters. Since it satisfy theorem 2.15 it can also be represented as an $|I| \times |L|$ triple array. Checking the parameters gives that it is a $TA(12, 3, 3, 1, 3 : 4 \times 9)$.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>
<i>a</i>	4	5	6	8	9	7	11	12	10
<i>b</i>	1	11	9	12	10	2	3	7	8
<i>c</i>	10	1	12	2	5	11	4	3	6
<i>d</i>	7	8	1	4	2	6	9	5	3

Chapter 3

Results

The intension of this chapter is to present what is known about triple arrays, and in order to do so we will define a new type of design called balanced grid. Almost all of the results are from McSorley et al [24] and McSorley [25], and that is why the results are formulated for triple arrays. At the end of the chapter we present some of our own results.

3.1 Triple arrays

A triple array is a double array so let us start with the fundamental properties.

Lemma 3.1. *Any $DA(v, k, \lambda_{rr}, \lambda_{cc} : r \times c)$ satisfies*

1. $vk = rc$;
2. $\lambda_{rr}(r - 1) = c(k - 1)$;
3. $\lambda_{cc}(c - 1) = r(k - 1)$;
4. $\lambda_{rr}r(r - 1) = \lambda_{cc}c(c - 1)$.

Proof. We have already proven (1) by double counting in remark 2.5. Next we will use that theorem 2.6 tells us that both rows and columns in a double array are duals of BIBD's. Equation (2) follows directly if we take the parameters from the dual of the rows in corollary 2.7 and plug them into the fundamental equation for BIBD's, that is equation (2) in theorem A.1. Equation (3) is the corresponding result for the dual of the columns. We get equation (4) by combining (2) and (3). \square

Lemma 3.2. *In any $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$,*

$$\lambda_{rc} = k.$$

TOMAS NILSON

Proof. Write I_{ij} for the intersection of row i with column j . If x is an entry on row i_1 , the binary property gives that x belongs to k of the sets I_{i_1j} . Since x is in k rows, we have that x is in k^2 of the sets I_{ij} . As there are v different entries, the total number of entries in the I_{ij} 's is k^2v . But each I_{ij} is a λ_{rc} set and there are rc of them. So $k^2v = rc\lambda_{rc}$. Since $vk = rc$ from theorem 3.1, the result follows. \square

How few symbols can there be in a $r \times c$ triple array? The answer of that question corresponds to the important inequality A.7 for BIBD's by Fisher.

Theorem 3.3. *Any triple array $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ satisfies,*

$$v \geq r + c - 1.$$

Proof. Suppose that \mathcal{D} is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$, and let symbols, rows and columns be the first, second and third constraints respectively. From definition 2.24 we know that $N_{21}N_{21}^T = (c - \lambda_{rr})I_r + \lambda_{rr}J_r$ and $N_{31}N_{31}^T = (r - \lambda_{cc})I_c + \lambda_{cc}J_c$. The same definition says that $N_{21}N_{31}^T = kJ_{r,c}$ and $N_{31}N_{21}^T = kJ_{c,r}$. We define the $(r + c) \times v$ matrix \mathcal{A} by

$$\mathcal{A} = \begin{bmatrix} N_{21} \\ N_{31} \end{bmatrix}$$

Then $\mathcal{A}\mathcal{A}^T$ is an $(r + c) \times (r + c)$ matrix, satisfying

$$\begin{aligned} \mathcal{A}\mathcal{A}^T &= \begin{bmatrix} N_{21} \\ N_{31} \end{bmatrix} \begin{bmatrix} N_{21}^T & | & N_{31}^T \end{bmatrix} = \begin{bmatrix} N_{21}N_{21}^T & | & N_{21}N_{31}^T \\ N_{31}N_{21}^T & | & N_{31}N_{31}^T \end{bmatrix} \\ &= \begin{bmatrix} (c - \lambda_{rr})I_r + \lambda_{rr}J_r & | & kJ_{r,c} \\ kJ_{c,r} & | & (r - \lambda_{cc})I_c + \lambda_{cc}J_c \end{bmatrix} \end{aligned}$$

If we can show that $\mathcal{A}\mathcal{A}^T$ has $rank$ $r + c - 1$. Then $r + c - 1 = rank(\mathcal{A}\mathcal{A}^T) \leq rank(\mathcal{A}) \leq v$, and we are done.

$$\mathcal{A}\mathcal{A}^T = \left[\begin{array}{cccc|cccc} c & \lambda_{rr} & \cdots & \lambda_{rr} & k & k & \cdots & k \\ \lambda_{rr} & c & \cdots & \lambda_{rr} & k & k & \cdots & k \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{rr} & \lambda_{rr} & \cdots & c & k & k & \cdots & k \\ \hline k & k & \cdots & k & r & \lambda_{cc} & \cdots & \lambda_{cc} \\ k & k & \cdots & k & \lambda_{cc} & r & \cdots & \lambda_{cc} \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\ k & k & \cdots & k & \lambda_{cc} & \lambda_{cc} & \cdots & r \end{array} \right]$$

Subtract column 1 from columns 2, 3, \dots , r and column $r + 1$ from columns $r +$

CHAPTER 3. RESULTS

$2, r + 3, \dots, r + c$, and we get

$$\left[\begin{array}{ccccc|ccccc} c & \lambda_{rr} - c & \lambda_{rr} - c & \cdots & \lambda_{rr} - c & k & 0 & 0 & \cdots & 0 \\ \lambda_{rr} & c - \lambda_{rr} & 0 & \cdots & 0 & k & 0 & 0 & \cdots & 0 \\ \lambda_{rr} & 0 & c - \lambda_{rr} & \cdots & 0 & k & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{rr} & 0 & 0 & \cdots & 0 & k & 0 & 0 & \cdots & 0 \\ \lambda_{rr} & 0 & 0 & \cdots & c - \lambda_{rr} & k & 0 & 0 & \cdots & 0 \\ \hline k & 0 & 0 & \cdots & 0 & r & \lambda_{cc} - r & \lambda_{cc} - r & \cdots & \lambda_{cc} - r \\ k & 0 & 0 & \cdots & 0 & \lambda_{cc} & r - \lambda_{cc} & 0 & \cdots & 0 \\ k & 0 & 0 & \cdots & 0 & \lambda_{cc} & 0 & r - \lambda_{cc} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \vdots \\ k & 0 & 0 & \cdots & 0 & \lambda_{cc} & 0 & 0 & \cdots & 0 \\ k & 0 & 0 & \cdots & 0 & \lambda_{cc} & 0 & 0 & \cdots & r - \lambda_{cc} \end{array} \right]$$

Add rows $2, 3, \dots, r$ to row 1, and rows $r + 2, r + 2, \dots, r + c$ to row $r + 1$. The $(1, 1)$ entry becomes $c + (r - 1)\lambda_{rr}$, and this equals ck by lemma 3.1. Similarly, the $(r + 1, r + 1)$ entry equals rk .

$$\left[\begin{array}{ccccc|ccccc} ck & 0 & 0 & \cdots & 0 & rk & 0 & 0 & \cdots & 0 \\ \lambda_{rr} & c - \lambda_{rr} & 0 & \cdots & 0 & k & 0 & 0 & \cdots & 0 \\ \lambda_{rr} & 0 & c - \lambda_{rr} & \cdots & 0 & k & 0 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_{rr} & 0 & 0 & \cdots & 0 & k & 0 & 0 & \cdots & 0 \\ \lambda_{rr} & 0 & 0 & \cdots & c - \lambda_{rr} & k & 0 & 0 & \cdots & 0 \\ \hline ck & 0 & 0 & \cdots & 0 & rk & 0 & 0 & \cdots & 0 \\ k & 0 & 0 & \cdots & 0 & \lambda_{cc} & r - \lambda_{cc} & 0 & \cdots & 0 \\ k & 0 & 0 & \cdots & 0 & \lambda_{cc} & 0 & r - \lambda_{cc} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & & \ddots & \vdots \\ k & 0 & 0 & \cdots & 0 & \lambda_{cc} & 0 & 0 & \cdots & 0 \\ k & 0 & 0 & \cdots & 0 & \lambda_{cc} & 0 & 0 & \cdots & r - \lambda_{cc} \end{array} \right]$$

If we delete row $r + 1$ and column $r + 1$ we get an $(r + c - 1) \times (r + c - 1)$ matrix with determinant $ck(c - \lambda_{rr})^{r-1}(r - \lambda_{cc})^{c-1}$. Since no rows or columns are repeated, it follows trivially that both $r - \lambda_{cc}, c - \lambda_{rr} > 0$, and the determinant is non-zero. Then \mathcal{AA}^T has *rank* at least $r + c - 1$, and since row 1 and row $r + 1$ in the last notated matrix are identical, it cannot have *rank* $r + c$. So \mathcal{AA}^T has *rank* $r + c - 1$ as required. \square

What about small arrays? We have seen triple arrays with replication number $k = 3$ and realize that k has to be greater than 1 since two rows intersects in $\lambda_{rr} > 0$.

Observation 3.4. There exists no triple array with $k = 2$.

Proof. Suppose $k = 2 < r < c$. Lemma 3.1 (3) gives $\lambda_{cc} = \frac{r}{c-1}$, which implies that $c = r + 1$. Then (2) in the same lemma gives $\lambda_{rr} = \frac{r+1}{r-1}$, and this is an integer only if $r = 2$ or $r = 3$. So $r = 3$ and $c = 4$. But there exists no 3×4 triple array by the exhaustive search in [24]. \square

Which is the smallest triple array? Suppose $r = 4 < c$. Then $k = 3$ and lemma 3.1 (1) gives $3v = 4c$, so $c \equiv 0 \pmod{3}$. Combining this with (3) $\lambda_{cc} = \frac{8}{c-1}$, gives $c = 9$. So $TA(12, 3, 3, 1, 3, : 4 \times 9)$ from example 2.17 has the minimal number of rows. If we instead compare the number of symbols the smallest is $TA(10, 3, 3, 2, 3 : 5 \times 6)$, an example can be found in the introduction.

3.1.1 Triple arrays with $v = r + c - 1$

All known triple arrays but one belongs to this class and we shall see that they have a relation to SBIBD's. It was Agrawal [4] that pointed out this connection when he constructed triple arrays from SBIBD's. However, his construction method 4.20 which we are going to study later is still unproved, and the results here concerning the relation are due to McSorley et al [24]. We will need a lemma on parameters.

Lemma 3.5. *Suppose \mathcal{A} is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with $v = r + c - 1$. Then*

$$\lambda_{cc} = r - \lambda_{rc} = v - 2c + \lambda_{rr} + 1.$$

Proof. From (1) in lemma 3.5 we have

$$r(c - 1) = rc - r = vk - r = (r + c - 1)k - r = r(k - 1) + k(c - 1),$$

so $r(k - 1) = (r - k)(c - 1)$ and then from (3)

$$\lambda_{cc} = \frac{r(k - 1)}{c - 1} = \frac{(r - k)(c - 1)}{c - 1} = r - k = r - \lambda_{rc}$$

by lemma 3.2.

We will use lemma 3.5 for the second equality too. Note that

$$\begin{aligned} \lambda_{cc} - \lambda_{rr} &= r - k - \lambda_{rr} \stackrel{(2)}{=} r - k - \frac{c(k-1)}{r-1} = \frac{(r-k)(r-1) - c(k-1)}{r-1} = \\ &= \frac{r(r-1) - k(r+c-1) + c}{r-1} = \frac{r(r-1) - kv + c}{r-1} \stackrel{(1)}{=} \frac{r(r-1) - rc + c}{r-1} = \\ &= \frac{r(r-1) - c(r-1)}{r-1} = r - c = r + c - 1 - 2c + 1 = v - 2c + 1. \end{aligned}$$

\square

CHAPTER 3. RESULTS

Now we are ready to look at the converse of Agrawals method, it was proved in McSorley et al [24].

Theorem 3.6. *Suppose \mathcal{A} is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with $v = r + c - 1$. Then there exists a $(v + 1, r, \lambda_{cc})$ SBIBD.*

Proof. Label the rows of \mathcal{A} as $R_i, i = 1, 2, \dots, r$ and the columns with $C_j, j = r + 1, r + 2, \dots, r + c$. The $r + c = v + 1$ indices of these labels will be our point set $\{1, 2, \dots, r + c\}$ when we construct a design \mathcal{D} . It has $v + 1$ blocks, one for each of the v symbols s in \mathcal{A}

$$B_s = \{i : s \notin R_i\} \cup \{j : s \in C_j\} \quad s = 1, 2, \dots, v$$

and the block

$$B_0 = \{1, 2, \dots, r\}.$$

The block size is constant since any given s does not occur in $r - k$ rows, and does occur in k columns in \mathcal{A} , so $|B_s| = (r - k) + k = r$.

\mathcal{D} is equireplicate since any point i will be in a B_s when s does not occur in row R_i in \mathcal{A} . This happens in $v - c$ cases, and in the block B_0 . So i will be in $v - c + 1 = r$ blocks. A point j occurs in different B_s for each of the r symbols in the column C_j of \mathcal{A} .

It remains to show that \mathcal{D} is balanced and we have three cases of unordered pairs.

(i_1, i_2) : Given a pair of rows R_{i_1}, R_{i_2} in \mathcal{A} , the sieve principle gives that $|((R_{i_1} \cup R_{i_2})^C)| = v - 2c + \lambda_{rr}$. Since the pair (i_1, i_2) also occurs in B_0 , it will be in $v - 2c + \lambda_{rr} + 1$ blocks in \mathcal{D} and lemma 3.5 gives that $v - 2c + \lambda_{rr} + 1 = \lambda_{cc}$.

(j_1, j_2) : A pair (j_1, j_2) will meet in λ_{cc} blocks of \mathcal{D} since $|C_{j_1} \cap C_{j_2}| = \lambda_{cc}$.

(i, j) : $|C_j \cap R_i^C| = r - \lambda_{rc}$. This is the number of blocks in \mathcal{D} where the pair (i, j) of symbols will meet. Lemma 3.5 gives that $r - \lambda_{rc} = \lambda_{cc}$.

So \mathcal{D} is a $(v + 1, r, \lambda_{cc})$ SBIBD. □

The component designs of the triple array will correspond to subdesigns of the constructed SBIBD. These subdesigns are defined in definitions A.3, A.9 and A.11.

Corollary 3.7. *Let \mathcal{A} be a triple array with $v = r + c - 1$ and let \mathcal{D} be the SBIBD constructed from \mathcal{A} in theorem 3.6, then*

1. *the dual design of the rows in \mathcal{A} is the complementary design of the derived design of \mathcal{D} with respect to B_0 ,*
2. *the dual design of the columns in \mathcal{A} is the residual design of \mathcal{D} with respect to B_0 .*

Proof. We know from lemma 2.7 that the dual designs of the rows and columns are BIBD's. From theorem 3.6 we know that \mathcal{D} has pointset $i = 1, 2, \dots, r$, $j = r + 1, r + 2, \dots, r + c$, and blocks

$$B_0 = \{1, 2, \dots, r\}, B_s = \{i : s \notin R_i\} \cup \{j : s \in C_j\} \quad s = 1, 2, \dots, v.$$

If we just look at the contribution from the rows we have blocks

$$B'_s = \{i : i \notin R_i\} \quad i = 1, 2, \dots, r, \quad s = 1, 2, \dots, v.$$

That means pointset B_0 and all blocks $B'_s = B_0 \setminus B_s$, and this is the complementary design of the derived design of \mathcal{D} with respect to B_0 , by definitions A.3 and A.9. The columns contributes with

$$B''_s = \{j : s \in R_j\} \quad j = r + 1, r + 2, \dots, r + c, \quad s = 1, 2, \dots, v.$$

We have pointset $\{1, 2, \dots, r + c\} \setminus B_0$ and all blocks $B''_s = B_s \setminus B_0$ and this is the residual design of \mathcal{D} with respect to B_0 by definition A.11. \square

Is the converse of theorem 3.6 true? This is probably the main question and it is known as *Agrawals conjecture*. We will look at it in chapter 4.3.

3.1.2 Triple arrays with $v > r + c - 1$

In 1976 Preece [32] asked for a $TA(35, 3, 5, 1, 3 : 7 \times 15)$. Probably since there are two suitable BIBD's satisfying definition 2.24, like, if M_1 and M_2 are the incidence matrices for the BIBD's respectively, then

$$M_1 M_2^T = 3J_{7,15}.$$

These BIBD's are not the residual design or the complement of a derived design of some SBIBD, and for many years, researchers didn't believe that there could be a triple array without a related SBIBD. However, McSorley et al [24] provided a computer generated example in their preprint.

Example 3.8. A $TA(35, 3, 5, 1, 3 : 7 \times 15)$. The only known triple array with $v > r + c - 1$.

31	1	18	16	7	10	5	3	4	2	33	14	19	15	12
26	32	1	2	29	30	28	20	27	11	5	34	3	8	4
1	17	13	9	3	4	21	22	6	35	25	5	24	2	23
6	27	33	28	16	13	35	30	15	10	9	26	12	17	29
16	12	23	32	34	21	15	33	24	22	11	10	8	25	20
21	22	28	24	25	19	7	14	18	29	27	23	26	30	31
11	7	8	14	13	32	20	6	34	18	19	17	35	31	9

CHAPTER 3. RESULTS

Yucas [46] studied the structure and gave a construction. The columndesign is a resolvable $(15, 35, 7, 3, 1)$ BIBD, a solution of the classical Kirkman schoolgirls problem. The rowdesign is a $(7, 35, 15, 3, 5)$ BIBD, a multiple of a $(7, 3, 1)$ SBIBD.

Remark 3.9. McSorley et al [24] also provided a list over parameters for a possible double or triple array with $v > r + c - 1$. They have not been able to construct any more triple arrays but in two of these cases they have found double arrays, $DA(99, 5, 18, 1 : 11 \times 45)$ and $DA(63, 5, 6, 3, : 15 \times 21)$.

3.2 Balanced grids

McSorley et al [24] introduced arrays called balanced grids, and the theory was further developed by McSorley [25]. Balanced grids are interesting for us since they in one important case offers an alternative definition for triple arrays. We do not quote all the results here, just enough to get the picture.

Definition 3.10. Let \mathcal{G} be a binary row-column design and define μ_{xy} to be the number of times that symbols x and y occur together in the same row or column of \mathcal{G} . If there is a constant μ such that $\mu_{xy} = \mu$ for every x and y then \mathcal{G} will be called a **balanced grid**.

Lemma 3.11. An $r \times c$ balanced grid based on v symbols satisfies,

$$\mu = \frac{rc(r+c-2)}{v(v-1)},$$

moreover, it will be equireplicate, with replication number

$$k = \frac{rc}{v}.$$

Proof. Let \mathcal{G} be an $r \times c$ balanced grid on a v -set V . Each of the $\binom{v}{2}$ pairs of symbols of V will occur μ times in the r rows and c columns of \mathcal{G} . Each row cover $\binom{c}{2}$ pairs, and each column $\binom{r}{2}$ pairs. Hence

$$\mu \binom{v}{2} = r \binom{c}{2} + c \binom{r}{2},$$

and the first result follows.

Take an $x \in V$ and suppose x occurs $k[x]$ times in \mathcal{G} . If x appears in the (i, j) cell, then x will form $c - 1$ pairs $\{x, y\}$ in row i , and $r - 1$ such pairs in column j , where $y \in V \setminus \{x\}$. So the symbol x in (i, j) forms $r + c - 2$ pairs. Since x occur in $k[x]$ cells it totally forms $(r + c - 2)k[x]$ pairs. On the other hand, there are $v - 1$ different pairs $\{x, y\}$ for any given x , so $k[x](r + c - 2) = \mu(v - 1)$. Therefore $k[x]$ is a constant k and doublecounting, like in lemma 3.1 gives $k = \frac{rc}{v}$. \square

We shall denote such a balanced grid by $BG(v, k, \mu : r \times c)$. Earlier we have seen some trivial examples of balanced grids. A Latin square is a $BG(v, v, 2v : v \times v)$, and a Youden square is a $BG(v, k, \mu : k \times v)$. Note that $v < r + c - 1$ in both of these examples, so they don't satisfy the inequality for triple arrays in theorem 3.3.

Theorem 3.12. *Any balanced grid $BG(v, k, \mu : r \times c)$ satisfies $v \leq r + c - 1$.*

Proof. Let \mathcal{G} be a $BG(v, k, \mu : r \times c)$. It is obvious that the theorem is true for the trivial BG with $r = c = 1$, so let us without loss of generality assume that $c \geq 2$. We will reuse the two incidence matrices N_{21}, N_{31} and the $(r + c) \times v$ matrix A , defined by

$$\mathcal{A} = \begin{bmatrix} N_{21} \\ N_{31} \end{bmatrix}$$

from the proof of theorem 3.3. This time we don't have that explicit information about the matrices, but we are going to look at the transpose $\mathcal{A}^T = [N_{21}^T N_{31}^T]$. The matrix $\mathcal{A}^T \mathcal{A} = [N_{21}^T N_{21} + N_{31}^T N_{31}] = [N_{12} N_{12}^T + N_{13} N_{13}^T]$ adds in cell (i, j) the number of rows and the number of columns that two symbols (s_i, s_j) both occurs in. Since \mathcal{G} is a balanced grid we have

$$\mathcal{A}^T \mathcal{A} = \begin{bmatrix} 2k & \mu & \mu & \mu & \dots & \mu \\ \mu & 2k & \mu & \mu & \dots & \mu \\ \mu & \mu & 2k & \mu & \dots & \mu \\ \vdots & \ddots & \ddots & \dots & \dots & \vdots \\ \mu & \mu & \mu & \dots & \dots & 2k \end{bmatrix}$$

This $v \times v$ matrix has determinant

$$(2k - \mu)^{v-1} (v\mu - \mu + 2k).$$

Suppose the determinant is zero. The factor $v\mu - \mu + 2k$ can not be zero since v, k and μ are positive. If $2k - \mu = 0$, we use lemma 3.11 that says $\mu = \frac{k(r+c-2)}{v-1}$, so $2 = \frac{r+c-2}{v-1}$ and $v = \frac{r+c}{2} \leq r + c - 1$ as claimed. In the case when $\det(\mathcal{A}^T \mathcal{A}) \neq 0$ we have

$$v = \text{rank}(\mathcal{A}^T \mathcal{A}) \leq \text{rank}(\mathcal{A}) \leq r + c.$$

We need to eliminate the possibility of $v = r + c$, so suppose $v = r + c$, then the expressions of lemma 3.11 becomes

$$\mu = \frac{rc(r+c-2)}{(r+c)(r+c-1)} \text{ and } k = \frac{rc}{r+c}.$$

As $\text{gcd}(r+c-2, r+c-1) = 1$, the first equality tells us that $r+c-1$ divides rc . The second gives that $r+c$ divides rc too, and since $\text{gcd}(r+c-1, r+c) = 1$, their product must divide rc . But $(r+c)(r+c-1) = r^2 + 2rc + c^2 - r - c > rc$, since $r^2 + c^2 > r + c$, so we have a contradiction. \square

3.2.1 Triple arrays and balanced grids

McSorley et al [24] proved the following relation between the arrays.

Theorem 3.13. *Any $TA(r+c-1, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$ is a $BG(r+c-1, k, k : r \times c)$.*

Proof. Let \mathcal{A} be a $TA(r+c-1, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$. From theorem 3.6 we know that there exists an associated SBIBD \mathcal{D} . Corollary 3.7 tells us that the column design of \mathcal{A} is the dual of the residual design of \mathcal{D} , and the row design is the dual of the complement to the derived design, both with respect to a block B_0 . Two symbols x and y in \mathcal{A} occur in column j if, and only if, blocks $B_x \setminus B_0$ and $B_y \setminus B_0$ both contains j . So the number of columns both x and y occurs in is

$$|(B_x \setminus B_0) \cap (B_y \setminus B_0)| = |B_x \cap B_y| - |B_x \cap B_y \cap B_0| = \lambda_{cc} - |B_x \cap B_y \cap B_0|.$$

Simulary, x and y both occur on row i if, and only if, blocks $B_0 \setminus B_x$ and $B_0 \setminus B_y$ both contains i . So the number of rows both x and y occurs in is

$$\begin{aligned} |(B_0 \setminus B_x) \cap (B_0 \setminus B_y)| &= |B_0| - |B_0 \cap B_x| - |B_0 \cap B_y| + |B_x \cap B_y \cap B_0| = \\ &= r - 2\lambda_{cc} + |B_x \cap B_y \cap B_0|. \end{aligned}$$

When we add these two we se that x and y occur together in $r - \lambda_{cc} = k$ rows and columns of \mathcal{A} , which makes it a $BG(r+c-1, k, k : r \times c)$. \square

Then, McSorley [25] proved the converse of theorem 3.13 and was able to summarize.

Theorem 3.14. *Let $v = r + c - 1$. Then every triple array is a $TA(v, k, c - k, r - k, k : r \times c)$ and every balanced grid is a $BG(v, k, k : r \times c)$, and they are equivalent.*

So if $v = r + c - 1$, we have an alternative way of proving that a design is a triple array. We also note that in this case the constant μ for balanced grids, corresponds to the constant μ for linked Graeco-latin designs in definition 2.10.

Remark 3.15. If $v = r + c - 1$ it is sufficient to write $TA(r \times c)$, since the other parameters then are obtainable.

3.3 More results

Here we present some of our own concluding results on triple arrays whith $v = r+c-1$. They offer alternative formulations, leaning on the properties of balanced grids.

Preece [32] says that the third constraint of a fully proper design of type $O : YZ(Q_1, Q_2, Q_3)$ has overall total balance, i.e. $Q_3 = T$ if

$$k_1 N_{31} N_{31}^T + k_2 N_{32} N_{32}^T = xI + yJ \quad (3.1)$$

for some x and y . But on the same page he asks for the triple array with $v > r + c - 1$ which does not satisfy equation (3.1). However, we have looked at this in the case where $v = r + c - 1$. Let us order the constraints symbols, rows and columns in that order as before.

Theorem 3.16. *An equireplicate binary row-column design with $v = r + c - 1$ is a triple array if and only if*

$$N_{21}^T N_{21} + N_{31}^T N_{31} = xI_v + xJ_v \quad x > 2.$$

Proof. Let \mathcal{D} be an equireplicate binary row-column design with $v = r + c - 1$ and denote the symbols s_1, s_2, \dots, s_v .

Each entry (i, j) in $N_{21}^T N_{21}$ gives us the number of rows in \mathcal{D} that contains both symbols s_i and s_j . The matrix $N_{31}^T N_{31}$ gives us the corresponding information for the columns. If the sum of these matrices is $xI + xJ$, it tells us that for each pair (s_i, s_j) , the sum of rows and columns that contain this pair is constant. So \mathcal{D} is a balanced grid and $x = \mu$ in BG-parameters. Since $v = r + c - 1$, theorem 3.14 gives that \mathcal{D} is a triple array and that $\lambda_{rc} = \mu$, so the $\lambda_{rc} = k > 2$ condition from observation 3.4 is satisfied.

A triple array with $v = r + c - 1$ is a balanced grid, so the sum of matrices is $\mu I + \mu J$, and since $x = \mu = \lambda_{rc}$ we know that $x > 2$. \square

After theorem 3.16 one might ask if there is an underlying BIBD with $xI + xJ$ as in theorem 1.3, that could help us to describe the structure of a triple array. This is not the case but there is such a design, closely related to BIBD's. We just have to skip the condition of constant block size.

Definition 3.17. *An arrangement of v symbols in b sets will be called a **pairwise balanced design** of index λ and type $(v : k_1, k_2, \dots, k_m)$ if each set contains k_1, k_2, \dots , or k_m symbols that are all distinct ($k_i \leq v, k_i \neq k_j$) and every pair of distinct symbols occurs in exactly λ sets of the design.*

Remark 3.18. Pairwise balanced designs are abbreviated PBD and an alternative parameter notation is $(v, b, r, \{k_1, \dots, k_m\}, \lambda)$.

We are now able to write a theorem which defines a triple array just in terms of structures.

Theorem 3.19. *Let \mathcal{D} be a row-column design with $v = r + c - 1$. Then \mathcal{D} is a triple array if, and only if,*

1. *the rows are the dual of a BIBD,*

CHAPTER 3. RESULTS

- 2. the columns are the dual of a BIBD,
- 3. the rows and columns are a PBD.

Proof. The first two conditions are well known from the proof of theorem 2.6. They also gives that \mathcal{D} is binary and equireplicate, but we have to prove the third condition. Suppose that \mathcal{D} is a triple array with $v = r + c - 1$. Let the rows and columns of \mathcal{D} be blocks in a design \mathcal{P} . Then \mathcal{P} has v points, $v+1$ blocks, two block sizes r, c and each symbol will occur in $2k$ blocks. Since theorem 3.14 gives that \mathcal{D} is a balanced grid we know that any two points s_1, s_2 will occur in $\mu = \lambda_{rc} = k$ of the blocks, so \mathcal{P} is balanced with parameters $PBD(v, v + 1, 2k, \{r, c\}, k)$.

Since balance for a PBD \mathcal{P} in this case with rows and columns as blocks, is equivalent with \mathcal{D} is a balanced grid, and since $v = r + c - 1$, we know that \mathcal{D} is a triple array. □

Example 3.20. A $PBD(12, 13, 6, \{4, 9\}, 3)$, extracted from the $TA(12, 3, 3, 1, 3, : 4 \times 9)$ in example 2.28.

	<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>	<i>G</i>	<i>H</i>	<i>I</i>
1	–	1	1	1	1	1	1	–	–	–	–	–	–
2	–	1	1	1	–	–	–	1	1	1	–	–	–
3	–	1	1	1	–	–	–	–	–	–	1	1	1
4	1	–	1	1	1	–	–	1	–	–	1	–	–
5	1	–	1	1	–	1	–	–	1	–	–	1	–
6	1	–	1	1	–	–	1	–	–	1	–	–	1
7	1	1	–	1	1	–	–	–	–	1	–	1	–
8	1	1	–	1	–	1	–	1	–	–	–	–	1
9	1	1	–	1	–	–	1	–	1	–	1	–	–
10	1	1	1	–	1	–	–	–	1	–	–	–	1
11	1	1	1	–	–	1	–	–	–	1	1	–	–
12	1	1	1	–	–	–	1	1	–	–	–	1	–

The PBD in the example above tells us that the symbol 12 occurs in rows a, b, c and columns C, D, H , in the triple array of example 2.28. But it does not give us the actual cells $(a, H), (b, D), (c, C)$, so we have a loss of information. Given a PBD with suitable parameters we do not know how to construct a triple array. But nevertheless, we want to know that such PBD's exists.

Observation 3.21. Let \mathcal{D} be a (v, k, λ) SBIBD. Then we can construct a $PBD(v - 1, v, 2(k - \lambda), \{k, v - k\}, k - \lambda)$.

Proof. Let N_1 be the incidence matrix of the residual design of \mathcal{D} with respect to a block B_0 and let N_2 be the incidence matrix of the complement of the derived design of \mathcal{D} with respect to B_0 . Then the matrix A

$$A = [N_2^T \mid N_1^T]$$

will be an incidence matrix of a PBD. We get a pointset of the $v - 1$ block indices of \mathcal{D} , and the v points of \mathcal{D} now labels the blocks in the PBD. We know from observation A.12 and A.13 that both N_1 and N_2 have $k - \lambda$ 1's in their columns so the number of 1's in each row of A will be a constant $2(k - \lambda)$. The number of 1's in a row of N_1 is k and in a row of N_2 it is $v - k$, and that gives the block sizes of the PBD. To see that the PBD is balanced take two blocks B_{s_1} and B_{s_2} in \mathcal{D} . The number of elements these blocks have in common in \mathcal{D} is

$$|(B_0 \setminus B_{s_1}) \cap (B_0 \setminus B_{s_2})| + |(B_{s_1} \setminus B_0) \cap (B_{s_2} \setminus B_0)| = k - \lambda.$$

(See the explicit calculation in the proof of theorem 3.13). This means that any two rows in A have $k - \lambda$ 1's in common, so the PBD is balanced. \square

Remark 3.22. Theorem 3.19 gives a definition of the design that are all structure. Even if the theorem is written for a row-column design it does not really require a representation, since it is just a description over the relations of the constraints. In that sense it is the opposite of the ordinary triple array definition, which is all about the properties of the representation. Of course is such a detailed definition suitable if one want to check details or do a computersearch as we have said before. The **RCDAO definition is in between but has the property adjusted orthogonality explicitivly given, which is important for staticians.

Chapter 4

Construction methods

In this chapter we are going to look at construction methods of triple arrays and their duals, the balanced Graeco-latin designs from definition 2.10. One might say that only one type of construction is proved, the infinite family of $q \times (q + 1)$ arrays where q is an odd primepower greater than three. Many authors have made constructions of this type and it is unclear who have contributed with what, but in 2005 Precce et al [35] gave a complete construction. Then we are going to look at Agrawals method where we construct triple arrays from SBIBD's. The method has one unproved step which gives rise to Agrawals conjecture.

4.1 Graeco-latin designs

Seberrys [39] provided a construction of Graeco-latin designs, dual to double arrays. In some cases the designs happen to be balanced, that is dual to a triple array but this is not given by the construction nor proved by Seberry. We have improved the construction so that it always gives balanced and linked Graeco-latin designs.

Most of the construction is done by letting the additive group of $GF(q)$ act on subsets of $GF(q)$. Since the theory about this is both standard and extensive we have choosen to put it in appendix A.1.2, which we often will refer to. Note that the sets Q of nonzero quadratic residues in $GF(q)$ and $R = GF(q) \setminus (Q \cup \{0\})$ from definition A.19 will play important roles. We will need the following lemma of our own.

Lemma 4.1. *Let $q > 3$ be an odd prime power. Then there exists primitive roots θ_1 and θ_2 in $GF(q)$ such that $(1 + \theta_1) \in Q$ and $(1 + \theta_2) \in R$.*

Proof. There is a standard result that there exists a primitive root θ in $GF(q)$, (see chapter 16.4 in Biggs [10]). Then θ^{-1} is a primitive root too since $(\theta^{-1})^i, i = 1, 2, \dots, q - 1$, is equivalent to $\theta^i, i = q - 1, q - 2, \dots, 1$, which are all the elements in $GF(q) \setminus \{0\}$. For an odd prime power $q > 3$, these two roots are distinct since

$\theta = \theta^{-1} \implies \theta^2 = 1$, which implies that $q = 3$. Note that $\theta \in R$.

If $(1 + \theta^{-1}) \in R$, then $(\theta + 1) \in \theta R \implies (\theta + 1) \in RR = Q$, and we are done.

If $(1 + \theta^{-1}) \in Q$, then $(\theta + 1) \in \theta Q \implies (\theta + 1) \in RQ = R$, and we are done. \square

We are ready to present our construction of balanced and linked Greaco-Latin designs, dual to triple arrays. It is built upon the construction by Seberry [39].

Construction 4.2. *Let $q > 3$ be an odd prime power and let x be a primitive root in $GF(q)$ such that, if $q \equiv 3 \pmod{4}$, then $1+x \in Q$, and if $q \equiv 1 \pmod{4}$, then $1+x \in R$. Then we may construct a balanced and linked Graeco-Latin design $\mathcal{D} = ((\mathcal{X} \times \mathcal{Y}), \mathcal{B})$ where $(\mathcal{X}, \mathcal{B})$ and $(\mathcal{Y}, \mathcal{B})$ are BIBD's. The elements in \mathcal{X} are the elements in $GF(q)$ and $\mathcal{Y} = \mathcal{X} \cup \{\infty\}$, where ∞ is an element with the property $\infty + i = \infty$, $\forall i \in GF(q)$. We construct \mathcal{D} by letting the additive group of $GF(q)$ act on the initial blocks B_0 and B'_0 , so \mathcal{D} consists of their orbits.*

1. If $q \equiv 3 \pmod{4}$ use the two blocks

$$B_0 = \{(0, \infty), (x^2, x^3), (x^4, x^5), \dots, (x^{q-1}, x)\}$$

$$B'_0 = \{(0, 0), (x, x^3), (x^3, x^5), \dots, (x^{q-2}, x)\}$$

2. If $q \equiv 1 \pmod{4}$ use the two blocks

$$B_0 = \{(0, \infty), (x, x^3), (x^3, x^5), \dots, (x^{q-2}, x)\}$$

$$B'_0 = \{(0, 0), (x^2, x), (x^4, x^3), \dots, (x^{q-1}, x^{q-2})\}$$

Proof. In the proof we will denote the set of elements in $GF(q)$ by G , and further use some short notation, like $Q_0 = Q \cup \{0\}$ and $R_\infty = R \cup \{\infty\}$.

First we will prove that $(\mathcal{X}, \mathcal{B})$ and $(\mathcal{Y}, \mathcal{B})$ are BIBD's. There are two cases.

- (1) $q \equiv 3 \pmod{4}$. We will also write $q = 4t - 1$.

The \mathcal{X} -parts of the starter blocks are Q_0 and R_0 , and they are both $(4t - 1, 2t, t)$ difference sets in $GF(q)$ by theorem A.28. So letting the additive group of $GF(q)$ act on the starter blocks, (we also say that we develop the starter blocks), gives two $(4t - 1, 2t, t)$ SBIBD's by theorem A.17, and they together form the BIBD $(\mathcal{X}, \mathcal{B})$ with parameters $(4t - 1, 2t, 2t)$.

For $(\mathcal{Y}, \mathcal{B})$ we have that the \mathcal{Y} -parts of the starter blocks are R_∞ and R_0 . We already know that R_0 is a $(4t - 1, 2t, t)$ difference set in $GF(q)$. Let us look at the R_∞ starter block, leaving the element ∞ out for the moment. Corollary A.27

CHAPTER 4. CONSTRUCTION METHODS

tells us that R is a $(4t - 1, 2t - 1, t - 1)$ difference set in $GF(q)$. If we develop R_0 and R , we get a $(4t - 1, (2t, 2t - 1), 2t - 1)$ design. When we put the element ∞ back into the R starter block, the block size becomes $2t$ here too, and the number of points becomes $|\mathcal{Y}| = 4t$. When we develop R_∞ and R_0 , pairs of the elements in $\mathcal{Y} \setminus \{\infty\}$ still meets in $2t - 1$ blocks, and ∞ will meet all of these elements $2t - 1$ times too, since ∞ is in all of the blocks developed from the R_∞ starter block, and the replication number for this part of the design is $|R| = 2t - 1$. So $(\mathcal{Y}, \mathcal{B})$ is a BIBD with parameters $(4t, 2t, 2t - 1)$.

(2) $q \equiv 1 \pmod{4}$. We will also write $q = 4t + 1$.

The \mathcal{X} -parts of the starter blocks are R_0 and Q_0 . Corollary A.30 gives that they are two $(4t + 1, 2t + 1, 2t + 1)$ supplementary difference sets in $GF(q)$. So by developing them both we get $(\mathcal{X}, \mathcal{B})$, which is a $(4t + 1, 2t + 1, 2t + 1)$ BIBD by theorem A.18. The \mathcal{Y} -parts of the starter blocks are R_∞ and R_0 . We know from corollary A.34 that R and R_0 are two supplementary difference sets with parameters $(4t + 1, (2t, 2t + 1), 2t)$. So far we have two block sizes but when we add ∞ to the smaller of these starter blocks we get $4t + 2$ points, and equal block size $2t + 1$. Pairs of elements in $\mathcal{Y} \setminus \{\infty\}$ still meets $2t$ times and ∞ will meet each of these points $2t$ times too, since ∞ is in every block developed from R_∞ where the replication number is $|R| = 2t$. So $(\mathcal{Y}, \mathcal{B})$ is a $(4t + 2, 2t + 1, 2t)$ BIBD.

Then we will prove that every pair of $\mathcal{X} \times \mathcal{Y}$ occurs exactly once in \mathcal{D} . Development of one of the starter blocks will give pairs (x, y) that are distinct. We see this if we look at the differences $y - x$ within the pairs. If $q \equiv 3 \pmod{4}$, then B_0 , the starter block containing ∞ has differences $x^{2m+1} + x_j - (x^{2m} + x_j) = x^{2m}(x - 1)$, $m = 1, 2, \dots, \frac{q-1}{2}$, and these products are all distinct in $GF(q)$. An investigation of the other starter blocks B'_0 will give the same result. We have to prove that the pairs are distinct when both starter blocks are developed, and will do this by showing that the differences $y - x$ are of different types in the two starter blocks.

(1) $q \equiv 3 \pmod{4}$.

In the starter block B_0 we have from x -denoted pairs differences $y - x$

$$x^{2m+1} - x^{2m} = x^{2m}(x - 1) \in \begin{cases} QQ = Q & \text{if } (x - 1) \in Q \\ QR = R & \text{if } (x - 1) \in R, \end{cases}$$

by lemma A.21 and notation by remark A.22. In B'_0 we have differences

$$\begin{aligned} x^{2m+3} - x^{2m+1} &= x^{2m+1}(x^2 - 1) = \\ &= x^{2m+1}(x + 1)(x - 1) \in \begin{cases} RQQ = R & \text{if } (x - 1) \in Q \\ RQR = Q & \text{if } (x - 1) \in R. \end{cases} \end{aligned}$$

We see that the two starter blocks will give different types of differences. The pairs obtained from $(0, \infty)$ and $(0, 0)$ are clearly distinct too, so there is no repetition of pairs.

(2) $q \equiv 1 \pmod{4}$.

In B_0 we have differences

$$\begin{aligned} x^{2m+3} - x^{2m+1} &= x^{2m+1}(x^2 - 1) = \\ &= x^{2m+1}(x+1)(x-1) \in \begin{cases} RRQ = Q \text{ if } (x-1) \in Q \\ RRR = R \text{ if } (x-1) \in R, \end{cases} \end{aligned}$$

and in B'_0 we have

$$\begin{aligned} x^{2m-1} - x^{2m} &= x^{2m-1}(1-x) = -x^{2m-1}(x-1) = \\ &= x^{2k-1}(x-1) \in \begin{cases} RQ = R \text{ if } (x-1) \in Q \\ RR = Q \text{ if } (x-1) \in R. \end{cases} \end{aligned}$$

So the pairs are distinct and since we have $q(\frac{q+1}{2} + \frac{q+1}{2}) = q(q+1)$ pairs, we have them all. At this stage we know that \mathcal{D} is a dual of a double array since it satisfies theorem 2.14.

Our next step is to prove that \mathcal{D} is balanced. This means that any given pair $x \in \mathcal{X}$ and $y \in \mathcal{Y}$ will meet in a constant number of blocks, not necessarily in the same pair of the ordered pair-elements. We will prove this by showing that the two starter blocks supplementary have all elements $d \in G_\infty$ as differences $d = y_0 - x_0$ a constant number λ of times, where a pair x_0, y_0 is in the same starter block. A given pair x, y with difference $d = y - x$, then will meet in all blocks B_i , where $d = y_0 - x_0$ and $i = x - x_0$ when the starter blocks are developed.

Let A and B be sets and let $k \in \mathbb{N}$. We locally defines $A - B$ to be a multiset $A - B = \{c : c = a - b, a \in A, b \in B\}$, where the differences c occurs with multiplicity. Addition is used for summations of differences, and kA is then a multiset with k copies of the elements in A . Let G denote the set of elements in $GF(q)$ where we calculate. There are two cases.

(1) $q \equiv 3 \pmod{4}$

The differences $y_0 - x_0$ in the two starter blocks are

$$R_\infty - Q_0 \text{ and } R_0 - R_0.$$

We calculate and count the differences

$$\begin{aligned} (R_\infty - Q_0) + (R_0 - R_0) &= \\ &= ((\{\infty\} - Q_0) + (R - Q_0)) + ((R - R) + (\{0\} - R_0) + (R - \{0\})) = \\ &= |Q_0|\{\infty\} + (R - Q_0 \cup R) + (-R_0) + R = \\ &= |Q_0|\{\infty\} + (R - G) + Q_0 + R = |Q_0|\{\infty\} + |R|G + G = |R_0|\{\infty\} + |R_0|G = |R_0|G_\infty. \end{aligned}$$

That $-R_0 = Q_0$ is given by lemma A.23 and A.21 using the notation in remark A.22. Since $-1 \in R$, then $-R_0 = RR_0 = Q_0$.

CHAPTER 4. CONSTRUCTION METHODS

(2) $q \equiv 1 \pmod{4}$.

Here we have differences

$$(R_\infty - R_0) + (Q_0 - R_0) = G_\infty - R_0 = |R_0|G_\infty.$$

In both cases, each difference $0, 1, \dots, q-1, \infty$ occur a constant number $\lambda = |R_0| = \frac{q+1}{2}$ times in the starter blocks. So any given pair x, y will meet in λ blocks. Hence \mathcal{D} is balanced.

Remark 4.3. At this stage we know by theorem 2.15 that \mathcal{D} is a dual of a triple array .

The last step is to prove that \mathcal{D} is linked so we will look at block intersections, and there are two cases.

(1) $q \equiv 3 \pmod{4}$.

The starter block B_0 has \mathcal{X}, \mathcal{Y} sets Q_0, R_∞ and B'_0 has R_0, R_0 . We have to consider tree subcases of intersections and begin by letting two distinct blocks B_i and B_j , both be developed from B_0 . These two blocks intersects in

$$\begin{aligned} |(Q_0 + i) \cap (Q_0 + j)| + |(R_\infty + i) \cap (R_\infty + j)| &= \\ &= \lambda_{Q_0} + |\{\infty\}| + \lambda_R = \frac{q+1}{4} + 1 + \frac{q-3}{4} = \frac{q+1}{2} \end{aligned}$$

points, since Q_0 and R are difference sets in $GF(q)$ by corollary A.28 and A.27. Corollary A.28 also gives the second subcase with two distinct blocks B'_i and B'_j , both developed from B'_0 , the number of points in which they intersects is

$$\begin{aligned} |(R_0 + i) \cap (R_0 + j)| + |(R_0 + i) \cap (R_0 + j)| &= \\ &= \lambda_{R_0} + \lambda_{R_0} = 2\frac{q+1}{4} = \frac{q+1}{2}. \end{aligned}$$

Then we calculate the intersection number for two blocks developed from different starter blocks. Let B_i be developed from B_0 , and let B'_j be developed from B'_0 . We will use theorem A.24 and corollary A.27 which tells us that Q and R are difference sets in $GF(q)$. We also use lemmas A.23 and A.21 to see that if $q \equiv 3 \pmod{4}$, then $Q = -R$.

$$\begin{aligned} |(Q_0 + i) \cap (R_0 + j)| + |(R_\infty + i) \cap (R_0 + j)| &= \\ &= |(Q_0 + i) \cap (G + j)| - |(Q_0 + i) \cap (Q + j)| + |(R + i) \cap (R_0 + j)| = \\ &= |Q_0| - |(Q + i) \cap (Q + j)| - |i \cap (Q + j)| + \\ &\quad + |(R + i) \cap (R + j)| + |(R + i) \cap j| = \\ &= |Q_0| - \lambda_Q - |i \cap (Q + j)| + \lambda_R + |(R + i) \cap j| = \\ &= |Q_0| - |i \cap (Q + j)| + |(R + i) \cap j| = \\ &= |Q_0| - |i \cap (-R + j)| + |(R + i) \cap j| = \\ &= |Q_0| - |(R + i) \cap j| + |(R + i) \cap j| = |Q_0| = \frac{q+1}{2}. \end{aligned}$$

TOMAS NILSON

So \mathcal{D} is linked when $q \equiv 3 \pmod{4}$, but there is one more case to consider.
(2) $q \equiv 1 \pmod{4}$.

The starter block B_0 has \mathcal{X}, \mathcal{Y} sets R_0, R_∞ and B'_0 has Q_0, R_0 . First subcase to check is the intersection of two distinct blocks B_i and B_j , both developed from B_0 . We will use corollary A.34 which tells us that R and R_0 are two supplementary difference sets in $GF(q)$. Their different cardinality doesn't matter here since we are only looking at intersections.

$$\begin{aligned} & |(R_0 + i) \cap (R_0 + j)| + |(R_\infty + i) \cap (R_\infty + j)| = \\ & = |(R_0 + i) \cap (R_0 + j)| + |(R + i) \cap (R + j)| + |\infty \cap (R_\infty + j)| = \\ & = \lambda_{RR_0} + |\infty \cap (R_\infty + j)| = \lambda_{RR_0} + 1 = \frac{q-1}{2} + 1 = \frac{q+1}{2}. \end{aligned}$$

Then we have the subcase when both blocks are developed from B'_0 . Here we use corollary A.30 which tells us that Q_0 and R_0 are two supplementary difference sets in $GF(q)$.

$$|(Q_0 + i) \cap (Q_0 + j)| + |(R_0 + i) \cap (R_0 + j)| = \lambda_{Q_0R_0} = \frac{q+1}{2}.$$

The last subcase to check is when one block B_i is developed from B_0 , and the other block B'_j is developed from B'_0 .

$$\begin{aligned} & |(R_0 + i) \cap (Q_0 + j)| + |(R_\infty + i) \cap (R_0 + j)| = \\ & = |(R_0 + i) \cap (Q_0 + j)| + |(R + i) \cap (R + j)| + |i \cap (Q_0 + j)| + |(R + i) \cap j| = \\ & = |(R + i) \cap G| + |i \cap (Q_0 + j)| + |i \cap (j - R)| = \\ & = |R| + |i \cap (Q_0 + j)| + |i \cap (j + R)| = \\ & = |R| + |i \cap G| = \frac{q-1}{2} + 1 = \frac{q+1}{2}. \end{aligned}$$

□

Example 4.4. We construct a balanced and linked Graeco-latin design \mathcal{D} by construction 4.2. Let $q = 5 \equiv 1 \pmod{4}$ and take primitive element $x = 2$. Then $x = 2, x^2 = 4, x^3 = 3, x^4 = 1$ in $GF(q)$, and $R = \{2, 3\}$, so $1 + x \in R$. The initial blocks are $B_0 = \{(0, \infty), (2, 3), (3, 2)\}$ and $B'_0 = \{(0, 0), (4, 2), (1, 3)\}$. We write the two initial blocks on the first line and then develop vertically in order to get the ten blocks.

$$\mathcal{D} = \begin{array}{l} B_0 = \{(0, \infty), (2, 3), (3, 2)\} \\ B_1 = \{(1, \infty), (3, 4), (4, 3)\} \\ B_2 = \{(2, \infty), (4, 0), (0, 4)\} \\ B_3 = \{(3, \infty), (0, 1), (1, 0)\} \\ B_4 = \{(4, \infty), (1, 2), (2, 1)\} \end{array} \left| \begin{array}{l} B'_0 = \{(0, 0), (4, 2), (1, 3)\} \\ B'_1 = \{(1, 1), (0, 3), (2, 4)\} \\ B'_2 = \{(2, 2), (1, 4), (3, 0)\} \\ B'_3 = \{(3, 3), (2, 0), (4, 1)\} \\ B'_4 = \{(4, 4), (3, 1), (0, 2)\} \end{array} \right.$$

4.1.1 The Δ -map

We can map the balanced Graeco-latin block designs from construction 4.2 to their duals, the triple arrays.

Construction 4.5. *We construct a $TA(2q, \frac{q+1}{2}, \frac{q+1}{2}, \frac{q-1}{2}, \frac{q+1}{2} : q \times (q+1))$ Δ from the Graeco-latin design \mathcal{D} in construction 4.2, by labeling the rows and columns by elements of \mathcal{X} and \mathcal{Y} respectively. As symbols we take the set of block indices s in \mathcal{D} .*

$$\Delta(x, y) = \begin{cases} s & \text{if } xy \text{ are developed from } B_0 \\ s' & \text{if } xy \text{ are developed from } B'_0. \end{cases}$$

If q is a prime, the additive group of $GF(q)$ is cyclic and this makes it easy to construct a triple array Δ directly from the starter blocks.

Observation 4.6. Let q be a prime and let the rows and columns in Δ be labeled in order $0, 1, \dots, q-1$ and $0, 1, \dots, q-1, \infty$, respectively. Then Δ can be developed as diagonal transversals, except for column ∞ , which is developed vertically. In all transversals the type is preserved.

Proof. If $(x, y) \in B_s$, then $((x+1), (y+1)) \in B_{s+1}$. This corresponds to $\Delta(x, y) = s$, then $\Delta(x+1, y+1) = s+1$ which is diagonal development and preserves the type s or s' . If $(x, \infty) \in B_s$, then $((x+1), \infty) \in B_{s+1}$, so in column ∞ the development is vertical, and the type is preserved here too. \square

Example 4.7. We want to map the Graeco-latin design \mathcal{D} from example 4.4 into a $TA(10, 3, 3, 2, 3 : 5 \times 6)$ by the Δ -map 4.5. But since 5 is a prime we can just look at the starter blocks $B_0 = \{(0, \infty), (2, 3), (3, 2)\}$ and $B'_0 = \{(0, 0), (4, 2), (1, 3)\}$. They tell us where 0 and 0' are mapped, and then we develop diagonally, as given in observation 4.6. For example $\Delta(0, \infty) = \Delta(2, 3) = \Delta(3, 2) = 0$.

$$\Delta = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & \infty \\ \hline 0 & 0' & 3 & 4' & 1' & 2 & 0 \\ 1 & 3 & 1' & 4 & 0' & 2' & 1 \\ 2 & 3' & 4 & 2' & 0 & 1' & 2 \\ 3 & 2' & 4' & 0 & 3' & 1 & 3 \\ 4 & 2 & 3' & 0' & 1 & 4' & 4 \end{array}$$

The B'_s -notation makes it easier to reason and prove, but when we present the design we often change the set of symbols.

Example 4.8. Take Δ from example 4.7 and map the dashed symbols by $s' \mapsto s+q$ in \mathbb{Z} , then we have

$$\Delta_1 = \begin{array}{cccccc} 5 & 3 & 9 & 6 & 2 & 0 \\ 3 & 6 & 4 & 5 & 7 & 1 \\ 8 & 4 & 7 & 0 & 6 & 2 \\ 7 & 9 & 0 & 8 & 1 & 3 \\ 2 & 8 & 5 & 1 & 9 & 4 \end{array}$$

4.2 Paley triple arrays

The Δ - map gives $q \times (q + 1)$ triple arrays from difference sets. Many authors like Potthoff [29], Agrawal [5], Preece [31] and Seberry [39] have worked with constructions like this, so it is a very popular method. Preece et al [35] gave a complete construction of these arrays. They call them *Paley triple arrays*, since the underlying $(v + 1, r, \lambda_{cc})$ -SBIBD's is a family of Hadamard designs, constructed by Paley in 1933. Note that several of the proofs in this section is somewhat alternative to the proofs in Preece et al [35], mainly because they do not use the concept of difference sets.

Definition 4.9. *If q is an odd prime power, a $TA(2q, \frac{q+1}{2}, \frac{q+1}{2}, \frac{q-1}{2}, \frac{q+1}{2} : q \times (q + 1)$ will be called a Paley triple array.*

The construction from Preece et al [35].

Construction 4.10. *Order the elements of $GF(q)$, say by $\{0 = w_0, 1 = w_1, w_2, \dots, w_{q-1}\}$, and let $GF(q)' = \{0' = w'_0, 1' = w'_1, w'_2, \dots, w'_{q-1}\}$ be a duplicate copy. For non-zero elements $a \neq 1$ and $b \neq -1$ in $GF(q)$ define the $q \times q$ matrix C_0 for $0 \leq i, j \leq q - 1$ by:*

$$C_0(i, j) = \begin{cases} w_i - \frac{w_i - w_j}{a} & \text{if } w_i - w_j \in Q \\ (w_i + \frac{w_i - w_j}{b})' & \text{if } w_i - w_j \in R_0 \end{cases}.$$

Let C be the $q \times (q + 1)$ matrix obtained by appending $(w_0, w_1, \dots, w_{q-1})$ to C_0 as column q , i.e. $C(i, q) = w_i$ for $i = 0, 1, \dots, q - 1$.

Remark 4.11. Notice that row i of C consists of

$$\{w_i - \frac{w_i - w_j}{a} : w_i - w_j \in Q\} \cup \{(w_i + \frac{w_i - w_j}{b})' : w_i - w_j \in R_0\} \cup \{w_i\}.$$

Column j for $0 \leq j < q$ consists of

$$\{w_j + \frac{a-1}{a}(w_i - w_j) : w_i - w_j \in Q\} \cup \{(w_j + \frac{b+1}{b}(w_i - w_j))' : w_i - w_j \in R_0\},$$

and is written in this way since we will have conditions on $w_i - w_j$. To see that the expression agrees with the construction we write

$$w_j + \frac{a-1}{a}(w_i - w_j) = w_j + w_i - w_j - \frac{w_i}{a} + \frac{w_j}{a} = w_i - \frac{w_i - w_j}{a}$$

and a similar check gives $w_j + \frac{b+1}{b}(w_i - w_j) = w_i + \frac{w_i - w_j}{b}$.

In order to construct Paley triple arrays certain choices have to be made, but every matrix C from construction 4.10 have the necessary fundamental properties.

CHAPTER 4. CONSTRUCTION METHODS

Proposition 4.12. *C as defined in construction 4.10 is a binary and equireplicate row-column design with replication number $\frac{q+1}{2}$.*

Proof. There is no repetition in any row i in columns $0, 1, \dots, q-1$, since $w_i - (w_i - w_{j_1})/a = w_i - (w_i - w_{j_2})/a \implies w_{j_1} = w_{j_2}$. Neither is the element w_i in column q repeated since $w_i - (w_i - w_j)/a = w_j \implies w_i - w_j = 0$, and $0 \notin Q$. Similarly, there are no repetitions in a column.

For the replication number, let $c \in GF(q)$. For each $w_i \in GF(q)$, there exists a unique $w_{j_i} \in GF(q)$ such that $w_i - (w_i - w_{j_i})/a = c$. If we let w_i go through all elements in $GF(q)$, so will $w_i - w_{j_i}$ but we only count the ones in Q . This gives $|(i, j) : C_0(i, j) = c| = |Q| = \frac{q-1}{2}$. Note that c also appears in column q so c appears $\frac{q-1}{2} + 1 = \frac{q+1}{2}$ times in C . The case when $c \in GF(q)'$ is similar. One arrives to $|(i, j) : C_0(i, j) = c| = |R_0| = \frac{q+1}{2}$. \square

For the not so fundamental properties we divide the formulation into two cases.

Theorem 4.13. *Suppose $q \equiv 3 \pmod{4}$, $q > 3$. Choose a and b such that $(a-1)(b+1) \in Q$ and if $(a-1) \in R$, then $ab \in Q$. Then C as defined in construction 4.10 is a Paley triple array.*

Proof. Here we have six different types depending on the choices of a and b . We will prove for one of these, let us call it type 1, but we define and comment on a type 2.

type 1: Let $a \in Q$, $b \in Q$ and $(a-1) \in Q$.

type 2: Let $a \in Q$, $b \in Q$ and $(a-1) \in R$.

We want to know what row i and column j in a type 1 array consists of.

Remark 4.14. Let $q \equiv 3 \pmod{4}$ and let $a, b \in Q$, $(a-1)(b+1) \in Q$ and $(a-1) \in Q$. We know from lemma A.23 that $(-1) \in R$, from lemma A.21 that $QQ = RR = Q$ and $QR = R$ by the notation in remark A.22. Since Q is a group under multiplication by lemma A.20 we will write $Q^{-1} = Q$ when we do the calculations with sets. Now, let's use this knowledge on the row and column expressions in remark 4.11.

If $(w_i - w_j) \in Q$, then row i in these positions consists of $w_i - \frac{Q}{Q} = w_i - QQ = w_i - Q = w_i + RQ = w_i + R$ and if we look at C we have $w_i + R_0$. If $(w_i - w_j) \in R_0$ we have $(w_i + \frac{R_0}{Q})' = (w_i + R_0Q)' = (w_i + R_0)'$.

Column j for $0 \leq j < q$ with $(w_i - w_j) \in Q$ consists of $w_j + \frac{Q}{Q}Q = w_j + Q$. If $(w_i - w_j) \in R_0$, then since $(a-1) \in Q$ here implicates that $(b+1) \in Q$ we get $(w_j + \frac{Q}{Q}R_0)' = (w_j + R_0)'$. Let's summarize:

The rows and columns in a type 1 array consists of

$$\begin{cases} \text{row } i & (w_i + R_0) \cup (w_i + R_0)' \\ \text{column } j & (w_j + Q) \cup (w_j + R_0)' \end{cases}$$

TOMAS NILSON

for $0 \leq j < q$.

The corresponding calculations for an array of type 2 gives

$$\begin{cases} \text{row } i & (w_i + R_0) \cup (w_i + R_0)' \\ \text{column } j & (w_j + R) \cup (w_j + Q_0)' \end{cases}$$

for $0 \leq j < q$.

Note that for all of these arrays we add a last column q , which consists of $Q \cup R \cup \{0\}$.

We are now ready to check that C satisfies the axioms in definition 2.3.

A1: Two distinct rows, say i_1 och i_2 intersects in $|(w_{i_1} + R_0) \cap (w_{i_2} + R_0)| + |(w_{i_1} + R_0) \cap (w_{i_2} + R_0)| = 2|[(w_{i_1} - w_{i_2}) + R_0] \cap R_0|$ symbols. From corollary A.28 we know, that if $q = 4t - 1$, then R_0 is a difference set in $GF(q)$ with $\lambda_{R_0} = t = \frac{q+1}{4}$. So, two distinct rows will intersect in $\frac{q+1}{2}$ symbols. In this case we have exactly the same for an array of type 2.

A2: Two distinct columns, say $j_1, j_2 < q$ intersects in $|(w_{j_1} + Q) \cap (w_{j_2} + Q)| + |(w_{j_1} + R_0) \cap (w_{j_2} + R_0)| = |[(w_{j_1} - w_{j_2}) + Q] \cap Q| + |[(w_{j_1} - w_{j_2}) + R_0] \cap R_0|$ symbols. Since Q is a difference set with $\lambda_Q = t - 1 = \frac{q-3}{4}$ and we know that $\lambda_{R_0} = \frac{q+1}{4}$ we get that two distinct columns intersects in $\frac{q-3}{4} + \frac{q+1}{4} = \frac{q-1}{2}$ symbols. Column q intersects column j , where $j < q$ in the $w_j + Q$ part, that is in $|Q| = 2t - 1 = \frac{q-1}{2}$ symbols too. For type 2 it's the same but one have to consider the difference sets R och Q_0 instead of Q och R_0 .

A3: If $i \neq j$ with $j < q$, then since R_0 is a difference set we know by corollary A.28, that the number of symbols row i intersect column j in, is

$$\begin{aligned} |(w_i + R_0) \cap (w_j + Q)| + |(w_i + R_0) \cap (w_j + R_0)| &= \\ &= |(w_i + R_0) \cap (w_j + G)| - |(w_i + R_0) \cap (w_j + R_0)| + \lambda_{R_0} = \\ &= |R_0| - \lambda_{R_0} + \lambda_{R_0} = |R_0| = \frac{q+1}{2}. \end{aligned}$$

If $i = j$ then row i intersects column i in $|(w_i + R_0) \cap (w_i + Q)| + |(w_i + R_0) \cap (w_i + R_0)| = 0 + |R_0| = \frac{q+1}{2}$ symbols. At last we have column q in C that intersects row i in $|w_i + R_0| = |R_0| = \frac{q+1}{2}$ symbols too. \square

Theorem 4.15. *Suppose $q \equiv 1 \pmod{4}$. Choose a and b such that $ab \in Q, (a - 1) \in Q$ and $(b + 1) \in R$. Then C as defined in construction 4.10 is a Paley triple array.*

Proof. Here we have two types:

type 1: Let $a \in Q$.

type 2: Let $a \in R$.

Let's determine what the rows and columns in a type 1 array consists of.

CHAPTER 4. CONSTRUCTION METHODS

Remark 4.16. Let $q \equiv 1 \pmod{4}$ and let $(a-1) \in Q, (b+1) \in R$ and $a \in Q$. Then we know by lemma A.21 that $b \in Q$, lemma A.23 gives that $(-1) \in Q$ and we know that $Q^{-1} = Q$ since Q is a group under multiplication. Baring this in mind, we look at the expressions in remark 4.11. If $(w_i - w_j) \in Q$, we get $w_i - \frac{Q}{Q} = w_i - QQ = w_i - Q = w_i + (-1)Q = w_i + QQ = w_i + Q$, which in C becomes $w_i + Q_0$. If $(w_i - w_j) \in R_0$ we get $(w_i + \frac{R_0}{Q})' = (w_i + R_0Q)' = (w_i + R_0)'$. For column j with $0 \leq j < q$, then if $(w_i - w_j) \in Q$, these cells consists by construction 4.10 of $w_j + \frac{Q}{Q}Q = w_j + Q$. If $(w_i - w_j) \in R_0$ we get $(w_j + \frac{R}{Q}R_0)' = (w_j + RQR_0)' = (w_j + RR_0)' = (w_j + Q_0)'$. Let's summarize:
The rows and columns in a type 1 array consists of

$$\begin{cases} \text{row } i & (w_i + Q_0) \cup (w_i + R_0)' \\ \text{column } j & (w_j + Q) \cup (w_j + Q_0)' \end{cases}$$

for $0 \leq j < q$.

The corresponding calculations for an array of type 2 gives:

$$\begin{cases} \text{row } i & (w_i + R_0) \cup (w_i + Q_0)' \\ \text{column } j & (w_j + R) \cup (w_j + R_0)' \end{cases}$$

for $0 \leq j < q$.

Column q in C consists in both cases of $Q \cup R \cup \{0\}$.

We check that C satisfies the axioms of definition 2.3.

A1: In both type 1 and 2, two distinct rows, say i_1 and i_2 intersects in $|(w_{i_1} + Q_0) \cap (w_{i_2} + Q_0)| + |(w_{i_1} + R_0) \cap (w_{i_2} + R_0)|$ symbols. Corollary A.30 gives that if $q = 4t + 1$, then Q_0 and R_0 are two supplementary difference sets, and the number of symbols the rows intersects is $\lambda_{Q_0R_0} = 2t + 1 = 2 \cdot \frac{q-1}{4} + 1 = \frac{q+1}{2}$.

A2: Two distinct columns, say $j_1, j_2 < q$ in a type 1 array intersects in $|(w_{j_1} + Q) \cap (w_{j_2} + Q)| + |(w_{j_1} + Q_0) \cap (w_{j_2} + Q_0)|$ symbols. Corollary A.33 gives that Q and Q_0 are supplementary difference sets and the columns intersects in $\lambda_{QQ_0} = 2t = 2 \cdot \frac{q-1}{4} = \frac{q-1}{2}$ symbols. Column q in C intersects column j , with $j < q$ in $w_j + Q$ or $w_j + R$ and that is $|Q| = |R| = 2t = \frac{q-1}{2}$ symbols too. The corresponding arguments holds for type 2, were we look at R and R_0 instead of Q and Q_0 .

A3: If C is of type 1 and $i \neq j$, with $j < q$ then row i intersects column j in

TOMAS NILSON

$|(w_i + Q_0) \cap (w_j + Q)| + |(w_i + R_0) \cap (w_j + Q_0)|$ symbols. We calculate:

$$\begin{aligned}
& |(w_i + Q_0) \cap (w_j + Q)| + |(w_i + R_0) \cap (w_j + Q_0)| = \\
& = |(w_i + Q) \cap (w_j + Q)| + |w_i \cap (w_j + Q)| + |(w_i + R_0) \cap (w_j + Q)| + |(w_i + R_0) \cap w_j| = \\
& = |(w_i + G) \cap (w_j + Q)| + |w_i \cap (w_j + Q)| + |(w_i + R_0) \cap w_j| = \\
& = |Q| + |w_i \cap (w_j + Q)| + |w_i \cap (w_j - R_0)| = \\
& = |Q| + |w_i \cap (w_j + Q)| + |w_i \cap (w_j + R_0)| = \\
& = |Q| + |w_i \cap (w_j + G)| = |Q| + 1 = \frac{q+1}{2},
\end{aligned}$$

where we used that $-R_0 = R_0$ by lemmas A.23 and A.21. If $i = j$, then row i intersects column j in $|Q| + 1$ symbols. The last column q intersects row i in $|w_i + Q_0| = |Q_0|$ symbols too. The same holds for a type 2 array but then one have to consider R instead of Q and R_0 instead of Q_0 . \square

If q is a prime, the additiv group in $GF(q)$ is cyclic and then we do not have to compute every entry in C explicitly, just like the Δ -map.

Proposition 4.17. *If q is a prime and $GF(q)$ is ordered by $\{0, 1, 2, \dots, q-1\}$, then C_0 has cyclic transversals. That is*

$$C_0(i+k, j+k) = C_0(i, j) + k.$$

Proof. Note that $(i+k) - (j+k) = i-j$, so all elements of a transversal will be in $GF(q)$ or all elements will be in $GF(q)'$. If $i-j \in Q$, then

$$C_0(i+k, j+k) = i+k - \frac{i-j}{a} = C_0(i, j) + k.$$

It is similar when $i-j \in R_0$. \square

So under the conditions of proposition 4.17 it's enough to compute the entries in row 0 in C_0 , and then develop diagonally.

Example 4.18. We will construct a $TA(14, 4, 4, 3, 4 : 7 \times 8)$. Theorem 4.13 tells us that construction 4.10 will give such a TA and how to choose a and b . In $GF(7)$ we have that $Q = \{1, 2, 4\}$. We choose $a = 2, b = 1$, then $(a-1)(b+1) \in Q$. Since $(a-1) \in Q$ the conditions of the theorem are satisfied and we just note that since $a, b \in Q$, the PTA will be of type 1.

Since 7 is a prime, we just calculate the entries of row 0.

$$\begin{aligned}
C_0(0, 0) &= \left(0 + \frac{0-0}{1}\right)' = 0' \quad \text{since } (0-0) \in R_0 \\
C_0(0, 1) &= \left(0 + \frac{0-1}{1}\right)' = 6' \quad \text{since } (0-1) = 6 \in R_0
\end{aligned}$$

CHAPTER 4. CONSTRUCTION METHODS

$$\begin{aligned}
 C_0(0, 2) &= (0 + 0 - 2)' = 5' \\
 C_0(0, 3) &= 0 - \frac{0 - 3}{2} = \frac{3}{2} = 3 \cdot 4 = 5 \\
 C_0(0, 4) &= (0 + 0 - 4)' = 3' \\
 C_0(0, 5) &= 0 - \frac{0 - 5}{2} = \frac{5}{2} = 5 \cdot 4 = 6 \\
 C_0(0, 6) &= 0 - \frac{0 - 6}{2} = \frac{6}{2} = 6 \cdot 4 = 3.
 \end{aligned}$$

We now develop row 0 in C_0 diagonally and add column q to the right where $w_i = i, i = 0, 1, \dots, 6$. The obtained array C is a $TA(14, 4, 4, 3, 4 : 7 \times 8)$.

$$\begin{array}{cccccccc}
 0' & 6' & 5' & 5 & 3' & 6 & 3 & 0 \\
 4 & 1' & 0' & 6' & 6 & 4' & 0 & 1 \\
 1 & 5 & 2' & 1' & 0' & 0 & 5' & 2 \\
 6' & 2 & 6 & 3' & 2' & 1' & 1 & 3 \\
 2 & 0' & 3 & 0 & 4' & 3' & 2' & 4 \\
 3' & 3 & 1' & 4 & 1 & 5' & 4' & 5 \\
 5' & 4' & 4 & 2' & 5 & 2 & 6' & 6
 \end{array}$$

4.3 Agrawals method

In 1966 Agrawal [4] provided a construction of triple arrays based on $(v+1, r, \lambda_{cc})$ SBIBD's. He could not prove the method, although he found no counterexample provided $r - \lambda_{cc} > 2$. However, this inequality does not exclude any triple arrays. The method gives that $r - \lambda_{cc}$ is equal to the replication number k for the array, and we know that $k > 2$ by observation 3.4. We note that the only SBIBD's with $r - \lambda_{cc} = 2$ are the $(7, 3, 1)$ SBIBD and its complement $(7, 4, 2)$, and they would correspond to non-existing 3×4 triple arrays.

Probably did not Agrawal have theorem 3.6 which says that we always can construct a SBIBD if we have a triple array with $v = r + c - 1$, or corollary 3.7 which establish the roles of the subdesigns. But in [1] he had already proved the following result.

Observation 4.19. Let \mathcal{D} be a (v, k, λ) SBIBD. Let N_1 denote the incidence matrix for the residual design of \mathcal{D} with respect to a block B_0 , and let N_2 denote the incidence matrix for the complement of the derived design of \mathcal{D} with respect to the same block B_0 . Then

$$N_1 N_2^T = (k - \lambda) J_{v-k, k}.$$

This tells us that it might be possible to merge this pair of BIBD's into a row-column design with adjusted orthogonality.

Construction 4.20. *Agrawals method.*

1. Let \mathcal{D} be a $(v+1, r, \lambda_{cc})$ SBIBD with $r - \lambda_{cc} > 2$ and label the blocks B_0, B_1, \dots, B_v .
2. Let \mathcal{D}_R be the residual design of \mathcal{D} with respect to B_0 , with blocks $B'_s = B_s \setminus B_0, s = 1, 2, \dots, v$. Label the elements $j = 1, 2, \dots, c$, where $c = v+1-r$.
3. Let N_2 denote the incidence matrix of the complement of the derived design of \mathcal{D} with respect to B_0 , with blocks $B''_s = B_0 \setminus B_s, s = 1, 2, \dots, v$. Label the elements $i = 1, 2, \dots, r$.
4. For each $s = 1, 2, \dots, v$, replace the entries 1 in column B''_s in N_2 by the elements of the block B'_s in any order, and let the remaining cells be undefined.
5. Rearrange the elements j within each column B''_s in defined cells so that each element of \mathcal{D}_R occurs exactly once in every row. Then we have an $r \times v$ array A where $A(i, s) = j$ in rc defined cells.
6. Map the defined triplets (i, s, j) from A to the $r \times c$ array C where $C(i, j) = s$. Then C is a $r \times c$ triple array.

Proof. Suppose that it is possible to rearrange the elements as in (5). Then there will be rc distinct pairs (i, j) amongst the rc defined triplets, so $C(i, j) = s$ can define an $r \times c$ row-column design. It is equireplicate since each block B''_s has a constant number $r - \lambda_{cc}$ of elements, and it is clearly binary. From observation A.12 we know that the residual design is a BIBD with parameters $(v+1-r, v, r, r - \lambda_{cc}, \lambda_{cc}) = (c, v, r, k, \lambda_{cc})$, if we write $k = r - \lambda_{cc}$. The complement of the derived design is by observation A.13 a BIBD with $(r, v, v - r + 1, r - \lambda_{cc}, v - 2r + \lambda_{cc} + 1) = (r, v, c, k, \lambda_{rr})$, if we write $\lambda_{rr} = v - 2r + \lambda_{cc} + 1$. So from the construction we know that both row-symbol and column-symbol structures are BIBD's so C is a double array.

The defined cells in the array A defines pairs (i, s) which are the same pairs as in N_2 , so the incidence matrix N_{is} is equal to N_2 . The corresponding holds for the defined pairs (j, s) . The incidence matrix N_{js} is equal to N_1 . Observation 4.19 gives

$$N_{js}N_{is}^T = N_1N_2^T = (r - \lambda_{cc})J_{c,r}$$

which tells us that i and j have adjusted ortogonality with respect to s by definition 2.21. Since i label the rows, j the columns and s the symbols in C , it is a triple array. \square

Example 4.21. Let us construct a triple array by constuction 4.20. We present the $(11, 5, 2)$ SBIBD \mathcal{D} by its incidence matix N . Without loss of generality we

CHAPTER 4. CONSTRUCTION METHODS

can write N so that B_0 consists of the first five elements.

$$N = \begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline \mathbf{1} & - & - & 1 & - & - & - & 1 & 1 & 1 & - & \\ \mathbf{1} & - & 1 & - & - & 1 & - & - & - & 1 & 1 & \\ \mathbf{1} & 1 & - & 1 & - & - & 1 & - & - & - & 1 & \\ \mathbf{1} & 1 & 1 & - & 1 & - & - & 1 & - & - & - & \\ \mathbf{1} & - & - & - & 1 & 1 & 1 & - & 1 & - & - & \\ - & 1 & - & - & 1 & - & - & - & 1 & 1 & 1 & \\ - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - & \\ - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & \\ - & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & \\ - & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & \\ - & - & 1 & - & - & - & 1 & 1 & 1 & - & 1 & \end{array}$$

Let N_1 denote the incidence matrix for the residual design \mathcal{D}_R of \mathcal{D} with respect to B_0 as in definition A.11. It is a $(6, 3, 2)$ BIBD. We label the elements $j = 1, 2, \dots, 6$.

$$N_1 = \begin{array}{c|cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & - \\ 3 & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - \\ 4 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 \\ 5 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - \\ 6 & - & 1 & - & - & - & 1 & 1 & 1 & - & 1 \end{array}$$

Let N_2 denote the incidence matrix for the complement to the derived design of \mathcal{D} with respect to B_0 . It is a $(5, 3, 3)$ BIBD. Label the elements $i = 1, 2, \dots, 5$.

$$N_2 = \begin{array}{c|cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 1 & 1 & - & 1 & 1 & 1 & - & - & - & 1 \\ 2 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - \\ 3 & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - \\ 4 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 \end{array}$$

Let the elements j from block B'_s of \mathcal{D}_R replace the 1's in column B''_s of N_2 for every $s = 1, 2, \dots, v$.

$$\begin{array}{c|cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 1 & 6 & & 1 & 2 & 6 & & & & 6 \\ 2 & 5 & & 2 & 3 & & 3 & 5 & 2 & & \\ 3 & & 3 & & 4 & 5 & & 6 & 6 & 5 & \\ 4 & & & 4 & & 4 & 5 & & 1 & 3 & 4 \\ 5 & 2 & 2 & 3 & & & & 4 & & 1 & 1 \end{array}$$

TOMAS NILSON

Rearrange the elements within columns so that every element occurs exactly once in every row. Then we have the array A where $A(i, s) = j$.

$$A = \begin{array}{c|cccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 5 & 6 & & 1 & 2 & 3 & & & & 4 \\ 2 & 1 & & 3 & 4 & & 6 & 5 & 2 & & \\ 3 & & 2 & & 3 & 5 & & 4 & 6 & 1 & \\ 4 & & & 2 & & 4 & 5 & & 1 & 3 & 6 \\ 5 & 2 & 3 & 4 & & & & 6 & & 5 & 1 \end{array}$$

Construct the array C by $C(i, j) = s$. It is a $TA(10, 3, 3, 2, 3 : 5 \times 6)$.

$$C = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 4 & 5 & 6 & 10 & 1 & 2 \\ 2 & 1 & 8 & 3 & 4 & 7 & 6 \\ 3 & 9 & 2 & 4 & 7 & 5 & 8 \\ 4 & 8 & 3 & 9 & 5 & 6 & 10 \\ 5 & 10 & 1 & 2 & 3 & 9 & 7 \end{array}$$

The rearrangement step in construction 4.20 is unproved, but Raghavarao and Nageswararao [37] claimed that they had proved it. They tried to reuse the proof of the corresponding theorem 1.10 for Youden squares. This corresponds to choosing a system of distinct representatives greedy in the preliminary array A for the sets B'_s in columns B''_s that have a defined cell in the first row and write this SDR_1 in these cells. Then choose another SDR_2 for the defined cells in the second row and so on until we have the rearranged array A in Agrawals method. However, this proof works for Youden squares because the rows there are complete. Every element will be a representative in each SDR, and Halls condition and theorem can be used after each SDR. But here the rows are incomplete, every element will not be a representative in each SDR and we have no control over what representatives that has been choosen. We cannot repeatedly claim that Halls condition is satisfied and therefore the proof does not hold. This flaw in the proof was pointed out by Wallis and Yucas [43]. They gave an explicit example of that the SDR's could not be choosen greedy as suggested, using a $(11, 5, 2)$ SBIBD which is unique up to isomorphism just like its subdesigns.

This gives rise to a conjecture.

Conjecture 4.22. *Agrawals conjecture.*

There is a $(v + 1, r, \lambda_{cc})$ SBIBD with $r - \lambda_{cc} > 2$ if and only if there is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$ with $v = r + c - 1$.

Yucas [46] wrote: *A proof of this fact that every symmetric balanced incomplete block design gives rise to a triple array would be very interesting. Such a proof would most likely depend on a very clever system of representatives type argument.*

Remark 4.23. Both Agrawal and Ragavarao-Nageswarerao interchange the meanings of residual and derived. In this thesis we use the common definitions as in A.9 and A.11.

4.3.1 Variations of the method and a generalization

Agrawals method can be expressed in several ways which can be of use when we construct arrays.

A variation of Agrawals method which we call *B-form* is to let the two BIBD's interchange roles. That is, for each $s = 1, 2, \dots, v$, replace the entries 1's in column B'_s in N_1 by the elements of the block B''_s in any order. Rearrange the elements within columns so that every element occurs exactly once in each row. Then we have an $c \times v$ array B with rc defined cells by $B(j, s) = i$. Map these triplets to the $r \times c$ array C by $C(i, j) = s$, which is a triple array.

Sterling and Wormald [42] used this variant to construct Graeco-latin designs from projective planes and we have seen one such design in example 2.28.

We have another variation of Agrawals method where we look directly at the SBIBD and put sets and SDR's in focus.

Construction 4.24. Let \mathcal{D} be a $(v+1, r, \lambda_{cc})$ SBIBD with blocks B_0, B_1, \dots, B_v . Let $R_i^C = \{s : x_i \in B_0 \text{ for } x_i \notin B_s\}$, $i = 1, 2, \dots, r$ and $R_j = \{s : x_j \notin B_0 \text{ for } x_j \in B_s\}$, $j = r+1, r+2, \dots, v+1$. The sets $R_i^C \cap R_j$ forms an incomplete blockdesign \mathcal{D}_I with parameters $(v, rc, (r - \lambda_{cc})^2, r - \lambda_{cc})$. Choose r SDR's for these blocks, one for each fixed $i = 1, 2, \dots, r$, $R_i^C \cap R_{r+1}, R_i^C \cap R_{r+2}, \dots, R_i^C \cap R_{v+1}$, such that if two representatives $s_{i_1, j_1}, s_{i_2, j_2}$ are equal, then $j_1 \neq j_2$. Write the representatives in an $r \times c$ array C by $C(i, j) = s_{ij}$ which is a triple array.

This is Agrawals method written in a different way and the only thing to prove is the parameters of D_I which we leave to the reader.

When using construction 4.24, it can be convenient to work with a $r \times c$ supermatrix where we write the entier set $R_i^C \cap R_j$ in the (i, j) cell. Then we choose the SDR's and thereby reduces the supermatrix to a matrix with single entries.

Example 4.25. We take the $(11, 5, 2)$ SBIBD from example 4.21 and construct a triple array. Label the rows in the incidence matrix $i = 1, 2, \dots, 5$, $j = 6, 7, \dots, 11$. We get sets of block indices $R_1^C = \{1, 2, 4, 5, 6, 10\}$ and $R_6 = \{1, 4, 8, 9, 10\}$, so $R_1^C \cap R_6 = \{1, 4, 10\}$ which we write in cell $(1, 6)$ in the supermatrix. Note that we here have labeled the columns $j = 6, 7, \dots, 11$.

	6	7	8	9	10	11
1	1, 4, 10	1, 2, 5	2, 4, 6	4, 5, 10	1, 5, 6	2, 6, 10
2	1, 4, 8	1, 3, 8	3, 4, 6	3, 4, 7	1, 6, 7	6, 7, 8
3	4, 8, 9	2, 5, 8	2, 4, 9	4, 5, 7	5, 7, 9	2, 7, 8
4	8, 9, 10	3, 5, 8	3, 6, 9	3, 5, 10	5, 6, 9	6, 8, 10
5	1, 9, 10	1, 2, 3	2, 3, 9	3, 7, 10	1, 7, 9	2, 7, 10

TOMAS NILSON

Then we choose five SDR's, one for each row so that if two representatives are equal, then they are in different columns.

$$C = \begin{array}{|c|c|c|c|c|c|} \hline & 1 & 2 & 4 & 10 & 5 & 6 \\ \hline & 4 & 1 & 6 & 3 & 7 & 8 \\ \hline & 8 & 5 & 2 & 4 & 9 & 7 \\ \hline & 9 & 8 & 3 & 5 & 6 & 10 \\ \hline & 10 & 3 & 9 & 7 & 1 & 2 \\ \hline \end{array}$$

Every construction method we have looked at so far deals with triple arrays corresponding to SBIBD's. But if we look at the proof of Agrawal's method we understand that the method works for any pair of suitable BIBD's. An example is the two component BIBD's in the 7×15 triple array which $v > r + c - 1$, which we looked at in section 3.1.2. Agrawal tells us how to find such BIBD's in the case $v = r + c - 1$, but his method will work in both cases provided that the rearrange step can be done. We believe that there exists more than one example of triple arrays which $v > r + c - 1$, so let us state the generalization.

Construction 4.26. *Agrawal's generalized method.*

Let N be the incidence matrix for an $(r, v, c, k, \lambda_{rr})$ BIBD D_1 with elements $i = 1, 2, \dots, r$ and blocks B_1, B_2, \dots, B_v . Let M be the incidence matrix for a $(c, v, r, k, \lambda_{cc})$ BIBD D_2 with elements $j = 1, 2, \dots, c$ and blocks B'_1, B'_2, \dots, B'_v . Suppose $k > 2; v \geq r + c - 1$ and that D_1 and D_2 are such that $NM^T = kJ_{r,c}$. Then:

1. For each $s = 1, 2, \dots, v$, replace the entries 1 in column B_s in N by the elements of the block B'_s in any order, and let the remaining cells be undefined.
2. Rearrange the elements j within each column B_s in defined cells so that every element of \mathcal{D}_2 occurs exactly once in every row. Then we have an $r \times v$ array A where $A(i, s) = j$ in rc defined cells.
3. Map the defined triplets (i, s, j) from A to the $r \times c$ array C by $C(i, j) = s$, which is a $TA(v, k, \lambda_{rr}, \lambda_{cc}, k : r \times c)$.

Chapter 5

More about the designs

In this chapter we have collected some short remarks about the designs, primarily about existence and a list over small and known triple arrays.

5.1 Isomorphic triple arrays

We need to know when two triple arrays are the same.

Definition 5.1. *Two triple arrays are isomorphic to one and another if one can be obtained from the other by some combinations of the operations*

1. *a permutation of rows,*
2. *a permutation of columns and*
3. *a permutations of the symbols.*

Not much has been written on isomorphic triple arrays. Phillips et al [27] have classified the $TA_{5 \times 6}$, which fall into seven isomorphism classes. Note that there is only one corresponding $(11, 5, 2)$ SBIBD up to isomorphism by Hall [18].

5.2 Existence of SBIBD's

The SBIBD's are well-studied objects and many constructions of them are known, (see Ionin and Shrikhande [21]), but the existence of a design with given parameters is nearly always undecided. We know that there is an infinite family of projective planes, that is $(v, k, 1)$ SBIBD's, but there is a conjecture that says: for any given $\lambda > 1$, the number of SBIBD's is finite. This is important for us since non-existence of SBIBD's excludes triple arrays. The most general non-existence result besides the fundamental equations in theorem A.1 is the Bruck-Ryser-Chowla theorem. A proof can be found in chapter 10.3 in Hall [18].

Theorem 5.2. (*BRC*). *If there exists a symmetric balanced incomplete block design with parameters (v, k, λ) , then:*

1. *if v is even, $k - \lambda$ must be a perfect square;*
2. *if v is odd, there must be integers x, y , and z , not all zero, such that*

$$x^2 = (k - \lambda)y^2 + (-1)^{(v-1)/2}\lambda z^2.$$

For example, there could be a $(22, 7, 2)$ SBIBD according to the fundamental equations but not by theorem 5.2, so thereby we know that there exists no $TA(21, 5, 10, 2, 5 : 7 \times 15)$.

We can also use result on possible subdesigns of SBIBD's. A *quasi-residual* design is a BIBD with the parameters of a residual design, i.e. $r = k + \lambda$. It is known that any quasi-residual design with $\lambda = 1, 2$ can be embedded in a SBIBD but for $\lambda > 2$ there exists non-embeddable designs. This is interesting since there exists a $DA(24, 6, 10, 3 : 9 \times 16)$ whose row-design is the $(16, 24, 9, 6, 3)$ BIBD given by Bhattacharya. He proved that this quasi-residual BIBD could not be embedded in a $(25, 9, 3)$ SBIBD, which we know can be constructed from a triple array. This means that there exists double arrays, that not can be permuted into a triple array by any permutations.

5.3 Triple arrays balanced for intersection

The row-column intersection sets in a triple array form a blockdesign. In the case where $v = r + c - 1$, it is isomorphic to the blockdesign D_I we defined in construction 4.24. This blockdesign is of great interest when we consider to prove Agrawal's method and one might ask if it is balanced. McSorley et al [24] have only found one such example, the D_I from $TA(10, 3, 3, 2, 3 : 5 \times 6)$ is a $(10, 30, 9, 3, 2)$ BIBD so the property seems to be rare. The triple array corresponds to a $(11, 5, 2)$ SBIBD and McSorley et al [24] presented some other SBIBD's with suitable parameters, for example $(56, 11, 2)$, $(66, 26, 10)$, $(149, 37, 9)$ and $(569, 72, 9)$, but they have not yet been able to construct any such BIBD's from them. However, we have looked at this and determined necessary properties for the SBIBD's. These properties which excludes all the suggested designs will be presented in a subsequent paper.

5.4 A list of small triple arrays

Parameters for the triple arrays $r \leq 12$, with $v = r + c - 1$ and their corresponding block designs. We have indicated where to our knowledge these arrays first appeared, in Potthoff [29], Agrawal [4] and McSorley et al [24].

TA	$BIBD_R$	$BIBD_C$	SBIBD	
$(v, k, \lambda_{rr}, \lambda_{cc}, \lambda_{rc} : r \times c)$	$(r, v, c, k, \lambda_{rr})$	$(c, v, r, k, \lambda_{cc})$	$(v + 1, r, \lambda_{cc})$	
(12, 3, 6, 1, 3 : 4 × 9)	(4, 12, 9, 3, 6)	(9, 12, 4, 3, 1)	(13, 4, 1)	Agr
(10, 3, 3, 2, 3 : 5 × 6)	(5, 10, 6, 3, 3)	(6, 10, 5, 3, 2)	(11, 5, 2)	Pott
(20, 4, 12, 1, 4 : 5 × 16)	(5, 20, 16, 4, 12)	(16, 20, 5, 4, 1)	(21, 5, 1)	Agr
(15, 4, 6, 2, 4 : 6 × 10)	(6, 15, 10, 4, 6)	(10, 15, 6, 4, 2)	(16, 6, 2)	Agr
(30, 5, 20, 1, 5 : 6 × 25)	(6, 30, 25, 5, 20)	(25, 30, 6, 5, 1)	(31, 6, 1)	Agr
(14, 4, 4, 3, 4 : 7 × 8)	(7, 14, 8, 4, 4)	(8, 14, 7, 4, 3)	(15, 7, 3)	Pott
(56, 7, 42, 1, 7 : 8 × 49)	(8, 56, 49, 7, 42)	(49, 56, 8, 7, 1)	(57, 8, 1)	McS
(18, 5, 5, 4, 5 : 9 × 10)	(9, 18, 10, 5, 5)	(10, 18, 9, 5, 4)	(19, 9, 4)	Agr
(24, 6, 10, 3, 6 : 9 × 16)	(9, 24, 16, 6, 10)	(16, 24, 9, 6, 3)	(25, 9, 3)	McS
(36, 7, 21, 2, 7 : 9 × 28)	(9, 36, 28, 7, 21)	(28, 36, 9, 7, 2)	(37, 9, 2)	McS
(72, 8, 56, 1, 8 : 9 × 64)	(9, 72, 64, 8, 56)	(64, 72, 9, 8, 1)	(73, 9, 1)	McS
(30, 7, 14, 3, 7 : 10 × 21)	(10, 30, 21, 7, 14)	(21, 30, 10, 7, 3)	(31, 10, 1)	McS
(90, 9, 72, 1, 9 : 10 × 81)	(10, 90, 81, 9, 72)	(81, 90, 10, 9, 1)	(91, 10, 1)	McS
(22, 6, 6, 5, 6 : 11 × 12)	(11, 22, 12, 6, 6)	(12, 22, 11, 6, 5)	(23, 11, 5)	Pott
(55, 9, 36, 2, 9 : 11 × 45)	(11, 55, 45, 9, 36)	(45, 55, 11, 9, 2)	(56, 11, 2)	McS
(44, 9, 24, 3, 9 : 12 × 33)	(12, 44, 33, 9, 24)	(33, 44, 12, 9, 3)	(45, 12, 3)	McS
(132, 11, 110, 1, 11 : 12 × 121)	(12, 132, 121, 11, 110)	(121, 132, 12, 11, 1)	(133, 12, 1)	McS

A more extensive list can be found in McSorley et al [24].

Appendix A

More preliminaries

In this section we have some definitions and results on 2-factor designs, difference sets and system of distinct representatives. Almost all of the results can be regarded as standard and we will give references for their proofs. In connection with these results we also have proved a few non-standard, but related lemmas that we will use in this thesis.

A.1 Preliminaries for 2-factor designs

The 2-factor designs we are interested in are block designs and BIBD's, both defined in definition 1.1. Here are their fundamental equations.

Theorem A.1. *Let \mathcal{D} be a block design with parameters (v, b, r, k) , then*

1. $vr = bk$. If \mathcal{D} further is balanced with index λ , then
2. $\lambda(v - 1) = r(k - 1)$.

A proof can be found in chapter 1.2 in Whitehead [44].

When are the structure of two designs the same.

Definition A.2. *Two designs $(\mathcal{X}, \mathcal{B})$ and $(\mathcal{X}', \mathcal{B}')$ are **isomorphic** if there is a bijective function $f : \mathcal{X} \rightarrow \mathcal{X}'$ such that f preserves incidence.*

A.1.1 Designs from designs

Given a design \mathcal{D} we can construct other designs.

Definition A.3. *Let $\mathcal{D} = (\mathcal{X}, \mathcal{B})$ be a block design with blocks $B_i : i = 0, 1, \dots, b-1$.*
1. The **complementary design** of \mathcal{D} has pointset \mathcal{X} and as blocks all $\mathcal{X} \setminus B_i$.

Observation A.4. Let \mathcal{D}^C be the complementary design of a (v, b, r, k, λ) BIBD \mathcal{D} . Then \mathcal{D}^C is a BIBD with parameters $(v, v - k, b - 2r + \lambda)$, provided that $b - 2r + \lambda > 0$.

APPENDIX A. MORE PRELIMINARIES

A proof can be found in chapter 2.1 in Whitehead [44]. The condition in observation A.4 only excludes trivial designs.

Observation A.5. In a BIBD we have that $b - 2r + \lambda = 0$ if, and only if, $k = v - 1$.

Proof. Substituting b in theorem A.1 (1) by b from the first equation gives $vr = k(2r - \lambda)$ and then $\lambda = \frac{r(2k-v)}{k}$. From (2) in the same theorem we have that $\lambda = \frac{r(k-1)}{v-1}$. We eliminate λ ,

$$\begin{aligned} \frac{r(2k-v)}{k} = \frac{r(k-1)}{v-1} &\iff (2k-v)(v-1) = k(k-1) \iff \\ &\iff 2kv - 2k - v^2 + v = k^2 - k \iff v^2 - 2kv + k^2 = v - k \iff \\ &\iff (v-k)^2 = v - k \iff v - k = 1. \end{aligned}$$

□

We defined the dual of a design in definition 1.4. What does a design has to satisfy in order to be a dual of a BIBD.

Proposition A.6. *Let \mathcal{E} be a design with n blocks on m points. If there exists positive constants c_1, c_2, c_3 such that*

1. *each block is incident with a constant number c_1 of points;*
2. *each point is incident with a constant number c_2 of blocks;*
3. *$c_2 < n$;*
4. *each unordered pair of distinct blocks intersects in a constant number c_3 of points.*

Then \mathcal{E} is a dual of a BIBD with parameters (n, m, c_1, c_2, c_3) .

Proof. Let \mathcal{D} be the dual design of \mathcal{E} . Definition 1.4 tells us that it has interchanged roles for points and blocks, so we check the dual statements of proposition A.6 in definition 1.1 of a BIBD. It is clear that \mathcal{D} has n points and m blocks. The dual statement of (1) is: *each point is incident with a constant number c_1 of blocks*, and it tells us that \mathcal{D} is equireplicate with replication number c_1 . The dual statement of (2) : *each block is incident with a constant number c_2 of points* tells us that \mathcal{D} has constant block size c_2 . By now we know that \mathcal{D} is a block design. That $c_2 < n$ then gives that \mathcal{D} is incomplete, and the dual statement of (3) tells us that \mathcal{D} is balanced with index c_3 . □

Can the dual of a BIBD be a BIBD too? There is an inequality that excludes most cases.

Theorem A.7. *Fisher's inequality.*

Let \mathcal{D} be a BIBD with b blocks and v points, then $v \leq b$.

A proof can be found in chapter 2.3 in Whitehead [44].

Theorem A.8. *Let \mathcal{D} be a BIBD. Then the dual design \mathcal{D}' is a BIBD if and only if \mathcal{D} is symmetric.*

A proof can be found in chapter 1 in Shrikhande [41]. It follows from theorem A.8 that any two blocks of a SBIBD \mathcal{D} intersects in a constant number μ of points. We then say that \mathcal{D} is **linked** and by theorem A.1 we can verify that for a (v, k, λ) SBIBD we have that the block intersection number $\mu = \lambda$.

Definition A.9. *Let \mathcal{D} be a SBIBD and let B_0 be a block of \mathcal{D} . The **derived design** of \mathcal{D} with respect to B_0 has pointset B_0 and as blocks all $B_i \cap B_0$, with $i \neq 0$.*

Observation A.10. Let \mathcal{D} be a (v, k, λ) SBIBD and let B_0 be a block of \mathcal{D} . The derived design of \mathcal{D} with respect to B_0 is a BIBD with parameters $(k, v - 1, k - 1, \lambda, \lambda - 1)$, provided that $\lambda \geq 2$.

A proof can be found in chapter 2.5 in Whitehead [44].

Definition A.11. *Let $\mathcal{D} = (\mathcal{X}, \mathcal{B})$ be a (v, k, λ) SBIBD and let B_0 be a block of \mathcal{D} . The **residual design** of \mathcal{D} with respect to B_0 has pointset $\mathcal{X} \setminus B_0$ and as blocks all $B_i \setminus B_0$ with $i \neq 0$.*

Observation A.12. Let \mathcal{D} be a (v, k, λ) SBIBD and let B_0 be a block of \mathcal{D} . The residual design of \mathcal{D} with respect to B_0 is a BIBD with parameters $(v - k, v - 1, k, k - \lambda, \lambda)$.

Proof. The pointset is of $|\mathcal{X} \setminus B_0| = v - k$ points. The block size is a constant $|B_i \setminus B_0| = k - \lambda$ since every pair of blocks in a SBIBD intersects in $\mu = \lambda$ points. The elimination of B_0 does not affect the remaining points so the residual is balanced with index λ and equireplicate with replication number $r = k$ since we started with a SBIBD. \square

Observation A.13. Let \mathcal{D} be a (v, k, λ) SBIBD and let B_0 be a block of \mathcal{D} . The complementary design of the derived design with respect to B_0 is a BIBD with parameters $(k, v - 1, v - k, k - \lambda, v - 2k + \lambda)$, provided that $v - 2k + \lambda > 0$.

To see that this is true, combine observations A.10, A.4 and theorem A.1.

APPENDIX A. MORE PRELIMINARIES

A.1.2 Difference sets

One way to construct a block design is to let a group act on one or more initial blocks so that the orbits of these blocks forms the design. In finite fields we often let the additive group act on difference sets.

Definition A.14. Let $(G, +)$ be an abelian group with $|G| = v$. Let B be a k -subset of G with the property that there exists a $\lambda \in \mathbb{Z}^+$ such that given any element $d \in G \setminus \{0\}$, there are exactly λ ordered pairs (x, y) of distinct elements $x, y \in B$ so that $x - y = d$. Then B is called a (v, k, λ) **difference set** in G .

Definition A.15. Let $(G, +)$ be an abelian group with $|G| = v$. Let B_1, B_2, \dots, B_h be a collection of k -subsets of G with the property that there exists a $\lambda \in \mathbb{Z}^+$ such that given any element $d \in G \setminus \{0\}$ there are in the set of ordered pairs $(x, y) \in B_i \times B_i, i = 1, 2, \dots, h$, exactly λ pairs so that $x - y = d$. Then the collection B_1, B_2, \dots, B_h is called a set of h **supplementary difference sets**.

Remark A.16. Let A be a subset of an abelian group G and let x be an element in G . By $A + x$ we mean the set $\{a + x : a \in A\}$.

Theorem A.17. Let B be a (v, k, λ) difference set in an abelian group G with $|G| = v$. Then the sets $B + x, x \in G$ form a (v, k, λ) SBIBD.

A relevant proof can be found in chapter 4.1 in Whitehead [44].

Theorem A.18. Let the k -subsets B_1, B_2, \dots, B_h of an abelian group G with $|G| = v$, be a collection of h supplementary difference sets with index λ . Then the sets $B_j + x, j = 1, 2, \dots, h, \forall x \in G$ form a (v, hv, hk, k, λ) BIBD.

The arguments for the proof of this theorem can be found in chapter 4.2 in Whitehead [44].

Definition A.19. Let θ be a primitive element in $GF(q)$. Then Q is the set of even powers of θ , also called **the set of nonzero quadratic residues** in $GF(q)$, and R is the set of odd powers of θ in $GF(q)$. We will denote $Q \cup \{0\}$ by Q_0 and $R \cup \{0\}$ by R_0 .

Lemma A.20. Let θ be a primitive element in $GF(q)$. Then Q is a cyclic group of order $\frac{q-1}{2}$ generated by θ^2 .

Proofs of this lemma and the following can be found in chapter 5.1 in Whitehead [44].

Lemma A.21. Let p be an odd prime. Then in $GF(p^n)$

1. if $x, y \in Q$ or $x, y \in R$, then $xy \in Q$,
2. if $x \in Q$ and $y \in R$, then $xy \in R$.

Remark A.22. When we have products as in lemma A.21 and only want to determine in what set the products lies, we will use short notation like $RR = QQ = Q$ and $QR = RQ = R$ as given in the lemma.

Lemma A.23. *In $GF(p^n)$. If $p^n \equiv 1 \pmod{4}$, then $(-1) \in Q$. If $p^n \equiv 3 \pmod{4}$, then $(-1) \in R$.*

A proof can be found in chapter 5.1 in Whitehead [44].

Theorem A.24. *Let $p^n = 4t - 1$, where p is a prime. Then the set Q of nonzero quadratic residues in $GF(p^n)$ is a $(4t - 1, 2t - 1, t - 1)$ difference set in $GF(p^n)$.*

A proof can be found in chapter 5.2 in Whitehead [44].

Example A.25. Let us look at the set Q in $GF(7)$. Theorem A.24 tells us that it is a $(7, 3, 1)$ difference set. We can find Q as the elements of the main diagonal of the multiplication table of $GF(7)^*$, or by taking the even powers of the primitive element 3, $3^2 = 2$, $3^4 = 4$, $3^6 = 1$. It is obvious that $|GF(7)| = 7$ and $|Q| = 3$. A difference table will make it clear that every nonzero difference of $GF(7)$ occurs exactly once in $Q = \{1, 2, 4\}$.

–	1	2	4	
1		6	4	
2	1		5	
4	3	2		

Theorem A.17 tells us how to construct a $(7, 3, 1)$ SBIBD by letting the additive group of $GF(7)$ act on Q . The seven blocks are $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{3, 4, 6\}$, $\{4, 5, 0\}$, $\{5, 6, 1\}$, $\{6, 0, 2\}$ and $\{0, 1, 3\}$.

Remark A.26. In the thesis we sometimes work with several difference sets and therefore use notation like λ_Q for the balance index λ corresponding to the difference set Q .

Corollary A.27. *Let $p^n = 4t - 1$, where p is a prime. Then the set R is a $(4t - 1, 2t - 1, t - 1)$ difference set in $GF(p^n)$.*

Corollary A.28. *Let $p^n = 4t - 1$, where p is a prime. Then Q_0 and R_0 are two $(4t - 1, 2t, t)$ difference sets in $GF(p^n)$.*

Theorem A.29. *Let $p^n = 4t + 1$, where p is a prime. Then Q and R are two supplementary difference sets in $GF(p^n)$ with parameters $(4t + 1, 2t, 2t - 1)$.*

A proof can be found in chapter 5.3 in Whitehead [44].

Corollary A.30. *Let $p^n = 4t + 1$. Then Q_0 and R_0 are two supplementary difference sets in $GF(p^n)$ with parameters $(4t + 1, 2t + 1, 2t + 1)$.*

APPENDIX A. MORE PRELIMINARIES

Remark A.31. In the proof of theorem A.29 it is common to let λ_1 denote the number of times an element in Q can be expressed as a difference of ordered pairs of elements in Q , and to let λ_2 denote the number of times the Q -element can be expressed as a difference of ordered pairs of elements in R . The same notation is used if we let the symbols Q and R change place.

Lemma A.32. *Let $p^n = 4t + 1$, where p is a prime. Then $\lambda_1 = t - 1$ and $\lambda_2 = t$.*

Proof. We will prove this in five steps. Let T_Q denote the difference tabel for Q .

- (1) $\lambda_1 + \lambda_2 = 2t - 1$.
- (2) If a row in T_Q has x quadratic residues, then the other rows have it too.
- (3) $1 - R$ has t quadratic residues and t non-quadratic residues.
- (4) $1 - Q$ has $t - 1$ quadratic residues and t non-quadratic residues.
- (5) T_Q has $\lambda_1 \cdot 2t$ quadratic residues and there is $2t(t - 1)$ of them. We can calculate that $\lambda_1 = t - 1$ and that $\lambda_2 = t$.

- (1) We know from the proof of theorem A.29 that $\lambda = \lambda_1 + \lambda_2 = 2t - 1$.
- (2) Suppose that T_Q has x quadratic residues in row $q_i - Q$, then there are x quadratic residues in row $\theta^2 q_i - Q$ because if q is a quadratic residue, $q = q_i - Q$ then we have a quadratic residues $\theta^2 q = \theta^2 q_i - \theta^2 Q$ since $QQ = Q$. From lemma A.20 we know that θ^2 generates Q , so we can get all rows in T_Q .
- (3) The differences $1 - GF(p^n)^*$ constitutes all elements $GF(p^n)$ except $1 \in Q$. Let us consider $1 - R$ as a list of $2t$ differences, $1 - \theta, 1 - \theta^3, \dots, 1 - \theta^{4t-1}$. Take a difference here, say $1 - \theta^{2i+1}$. If we multiply it with a non-quadratic residue $\theta^{2t-(2i+1)}$ we will get

$$\begin{aligned} \theta^{2t-(2i+1)}(1 - \theta^{2i+1}) &= \theta^{2(t-i)-1} - \theta^{2t} = 1 + \theta^{2(t-i)-1} = 1 - \theta^{2t}\theta^{2t-2i-1} = \\ &= 1 - \theta^{4t-2i-1} = 1 - \theta^{4t-(2i+1)}. \end{aligned}$$

Note that is $1 - \theta^{2i+1}$ is a quadratic residue, then $1 - \theta^{4t-(2i+1)}$ is a non-quadratic residues and inversely since $RQ = R$ och $RR = Q$. So we can pair up the differences in our list in t pairs, where every pair concists of a quadratic and a non-quadratic residue.

$$\begin{array}{ccc} 1 - \theta & \text{---} & 1 - \theta^{4t-1} \\ 1 - \theta^3 & \text{---} & 1 - \theta^{4t-3} \\ \vdots & \text{---} & \vdots \\ 1 - \theta^{2t-1} & \text{---} & 1 - \theta^{2t+1} \end{array}$$

So $1 - R$ gives t quadratic residues and t non-quadratic residues.

(4) There are $2t - 1$ non-quadratic residues in $GF(p^n) \setminus \{1\}$ and from (3) we know that t of them is in $1 - R$. Hence there are $t - 1$ quadratic residues in $1 - Q$ since we cannot not get $\{1\}$. So in row $1 - Q$ in T_Q there are $t - 1$ quadratic residues and from (2) we know that it holds for all rows. That there are t non-quadratic residues in $1 - Q$ is then easy to see.

(5) Since T_Q has $\lambda_1 \cdot 2t$ quadratic residues and there are $2t$ rows with $t - 1$ quadratic residues in every row we can calculate $\lambda_1 = t - 1$ and then $\lambda_2 = t$. \square

We will now define some supplementary difference sets with different cardinality. Since they are not standard results, we will prove them.

Theorem A.33. *Let $p^n = 4t + 1$, where p is a prime. Then Q and Q_0 are two supplementary difference sets in $GF(p^n)$ with parameters $(4t + 1, (2t, 2t + 1), 2t)$.*

Proof. We know by remark A.31 and lemma A.32 that ordered pairs of Q will have every difference $d \in Q$ $t - 1$ number of times and every $d \in R$ t times. If we look at the consequences of adding $\{0\}$ to the set Q , it will be $0 - Q = -Q = Q$ by lemma A.23 and $Q - 0 = Q$. So $\{0\}$ will contribute with all elements in Q twice and nothing more. So when we look at Q and Q_0 , the elements in R will occur as differences $2\lambda_2 = 2t$ times and the elements in Q will occur $2\lambda_1 + 2 = 2(t - 1) + 2 = 2t$ times too. Hence Q and Q_0 are two supplementary difference sets in $GF(q)$. \square

Corollary A.34. *Let $p^n = 4t + 1$. Then R and R_0 are two supplementary difference sets in $GF(p^n)$ with parameters $(4t + 1, (2t, 2t + 1), 2t)$.*

Proof. If we change the symbols Q and R in the proof of theorem A.33, it will work here too by remark A.31. \square

A.2 System of distinct representatives

Definition A.35. *Let A_1, A_2, \dots, A_n be sets. A system of distinct representatives (SDR) for these sets is a n -tuple (x_1, x_2, \dots, x_n) of elements with the properties*

- 1) $x_i \in A_i$, for $i = 1, 2, \dots, n$;
- 2) $x_i \neq x_j$, for $i \neq j$.

Definition A.36. *For any set $J \subseteq \{1, 2, \dots, n\}$ we define*

$$A(J) = \cup_{j \in J} A_j.$$

Theorem A.37. *(P. Hall 1935). A necessary and sufficient condition for the existence of an SDR for the collection of finite sets A_1, A_2, \dots, A_n is that*

$$|A(J)| \geq |J| \quad \text{for all } J \subseteq \{1, 2, \dots, n\}.$$

A proof can be found in chapter 6.2 in Cameron [11].

Index of notation

$GF(q)$	Galois field, a finite field with q elements	
Q	The set of nonzero quadratic residues in $GF(q)$	Definition A.19
Q_0	$Q \cup \{0\}$	
R	The set $GF(q) \setminus Q_0$	Definition A.19
∞	An element invariant under addition	Construction 4.2
BIBD	Balanced incomplete block design	Definition 1.1
SBIBD	Symmetric BIBD	Definition 1.1
λ_Q	Balance index λ corresponding to Q	Remark A.26
PBD	Pairwise balanced design	Definition 3.17
$J_{r,c}$	An $r \times c$ -matrix in which every entry is equal to 1	
N_{21}	An incidence matrix for constraints 2 and 1	Section 2.2
RCD	Row-column design	Definition 1.6
LS	Latin square	Definition 1.7
YS	Youden square	Definition 1.9
DA	Double array	Definition 2.3
TA	Triple array	Definition 2.3
BG	Balanced grid	Definition 3.10

Bibliography

- [1] AGRAWAL, HIRALAL: *On balanced block designs with two different number of replications*, Journal of the Indian Statistical Association, 1(1963), 145-51.

- [2] AGRAWAL, HIRALAL: *On the dual of BIB designs*, Calcutta Statistical Association Bulltin, 12(1963), 104-5.

- [3] AGRAWAL, HIRALAL: *Some generalizations of distinct representatives with applications to statistical designs*, The Annals for Math. Stat. 37 (1966), pp. 525-528.

- [4] AGRAWAL, HIRALAL: *Some methods of construction of designs for two-way elimination of heterogeneity*, Amer. stat. ass, (1966), no 61, pp 1153-1171.

- [5] AGRAWAL, HIRALAL: *Some systematic methods of construction o designs for two-way elimination of heterogeneity*, Calcutta Statist. Assoc. Bull., 15, (1966), 93-108.

- [6] H. L. AGRAWAL AND K. L. SHARMA: *On construction of two-way designs*, Journal of the Indian Statistical Association. Vol. 13, 1975, 1-31.

- [7] H. L. AGRAWAL AND K. L. SHARMA: *On Construction of Four-Dimensional Incomplete Block Designs*, Journal of the American Statistical Association, Vol. 73, No 364 (Dec., 1978). Theory and Methods Section.

BIBLIOGRAPHY

- [8] G. W. BEYNON: *Tournament and Duplicate Bridge*, Stuyvesant Press(1944).
- [9] G. W. BEYNON: *Bridge Director's Manual for Duplicate Games*, George Coffin, Publicher, Waltham (1955).
- [10] N. L. BIGGS: *Discrete Mathematics*, Oxford University Press, Revised edition (1989).
- [11] P. J. CAMERON: *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press 1994.
- [12] B. D. CAUSEY: *Some examples of multi-dimensional incomplete block designs*, The annals of Mathematical Statistics, Vol. 39, No 5(1968), 1577-1590.
- [13] FENG-SHUN CHAI: *A note on balanced generalized two-way elimination of heterogeneity designs*, Statistics & Probability Letters 29 (1996) 131-141.
- [14] FENG-SHUN CHAI: *A note on generalization of distinct representatives*, Statistics & Probability Letters 39 (1998) 173-177.
- [15] J. ECCLESTON AND K. RUSSEL: *Connectedness and orthogonality in multi-factor designs*, Biometrika, 62, 1975, pp. 341-345.
- [16] J. ECCLESTON AND K. RUSSEL: *Adjusted orthogonality in nonorthogonal designs*, Biometrika, 64, 1977, pp. 339-45.
- [17] J. A. ECCLESTON AND D. J. STREET: *Construction methods for adjusted orthogonal row-column designs*, Ars Combin. (1990), 28, 117-128.
- [18] MARSHALL HALL, JR: *Combinatorial Theory*, Second Edition, John Wiley& Sons, Inc. USA 1998.

TOMAS NILSON

- [19] T. N. HOBLYN, S. C. PEARCE AND G. H. FREEMAN: *Some Considerations in the Design of Successive Experiments in Fruit Plantations*, Biometrics, Vol. 10, No. 4 (Dec., 1954), 503-515.
- [20] F. W. LEVI: *Finite Geometrical Systems*, University of Calcutta, (1942).
- [21] Y. J. IONIN AND M. S. SHRIKHANDE: *Combinatorics of Symmetric Designs*, Cambridge University Press 2006.
- [22] J. A. JOHN AND J. A. ECCLESTON: *Row-column α -designs*, Biometrika (1986), 73, 2, pp. 301-6.
- [23] S. M. LEWIS AND A. M. DEAN: *On general balance in row-column designs*, Biometrika, 78, 3 (1991), pp. 595-600.
- [24] J. P. MCSORLEY, N. C. K. PHILLIPS, W. D. WALLIS AND J. L. YUCAS: *Double Arrays, Triple Arrays and Balanced Grids*, Designs, Codes and Cryptography, 35, 21-45, 2005.
- [25] JOHN P. MCSORLEY: *Double Arrays, Triple Arrays, and Balanced Grids with $v = r + c - 1$* , Designs, Codes and Cryptography, 37, 313-318, 2005.
- [26] S. C. PEARCE: *The use and classification of non-orthogonal designs*, J. R. statist. Soc. A126, (1963), 353-77.
- [27] N. C. K. PHILLIPS, D. A. PREECE AND W. D. WALLIS: *The seven classes of 5×6 triple arrays*, Discrete Mathematics 293 (2005), 213-218.
- [28] R. F. POTTHOFF: *Multi-Dimensional Incomplete Block Designs*, Institute of Statistics Mimeograph Series No. 211, University of North Carolina, Chapel Hill, N. C.(1958).
- [29] R. F. POTTHOFF: *Three-Factor Additive Designs More General Than the Latin Square*, Technometrics, Vol. 4, No 2(May., 1962), 187-208.

BIBLIOGRAPHY

- [30] D. A. PREECE: *Some Row and Column Designs for Two Sets of Treatments*, Biometrics, Vol. 22, No. 1(Mar., 1966), 1-25.
- [31] D. A. PREECE: *Some Balanced Incomplete Block Designs for Two Sets of Treatments*, Biometrics, Vol. 53, No. 3/4(Dec., 1966), 479-506.
- [32] D. A. PREECE: *NON-ORTHOGONAL GRAECO-LATIN DESIGNS*, Combinatorial mathematics, IV(Proc. Fourth Australian Conf., Univ. Adelaide, Adelaide 1975), pp. 7-16. Lecture Notes in Math., Vol. 560, Springer, Berlin, 1976.
- [33] D. A. PREECE: *ORTHOGONALITY AND DESIGNS: A TERMINOLOGICAL MUDDLE*, Utilitas Mathematica Vol. 12 (1977), pp. 201-223.
- [34] D. A. PREECE: *Multi-factor balanced block designs with complete adjusted orthogonality for all pairs of treatment factors*, Austral. J. Statist. 38(2), 1996, 223-230.
- [35] D. A. PREECE, W. D. WALLIS AND J. L. YUCAS: *Paley triple arrays*, Australasian Journal of Combinatorics v. 33 (2005), 237-246.
- [36] DAMARAJU RAGHAVARAO: *Constructions and Combinatorial Problems in Design of Experiments*, Dover Publications, INC 1988 (John Wiley & Sons, Inc 1971).
- [37] D. RAGHAVARAO AND G. NAGESWARARAO: *A note on a method of construction of designs for two-way elimination of heterogeneity*, Communications in statistics, 3(2), 197-199 (1974).
- [38] D. H. REES: *A generalisation of some Kirkman designs*, Utilitas Mathematica 32, 1987, 159-168.
- [39] SEBERRY, JENNIFER: *A note on orthogonal Graeco-Latin designs*, Ars Combinatoria, Vol. 8 (1979), pp. 85-94.

TOMAS NILSON

- [40] S. S. SHRIKHANDE: *Designs for two-way elimination of heterogeneity*, Annals of Mathematical Statistics, 22 (1951), 235-247.
- [41] MOHAN S. SHRIKHANDE AND SHARAD S. SANE: *Quasi-Symmetric Designs*, Cambridge University Press. USA 1991.
- [42] L. S. STERLING AND N. WORMALD: *A remark on the construction of designs for two-way elimination of heterogeneity*, Bull. Austral. Math. Soc., 14 (1976), 383-388.
- [43] W. D. WALLIS AND J. L. YUCAS: *Note on Agrawal's "Designs for Two-way Elimination of Heterogeneity"*, JCMCC 46 (2003), pp. 155-160.
- [44] C. A. WHITEHEAD: *The theory of designs*, University of London, External Publications 1996.
- [45] W. J. YODEN: *Use of incomplete block replications in estimating tobacco mosaic virus*, Contributions from Boyce Thompson Institute, 9(1937), 41-48.
- [46] JOSEPH L. YUCAS: *The Structure of a 7×15 Triple Array*, CONGRESSUS NUMERANTIUM 154 (2002), pp. 43-37.