Determining the elastic constants of paper from measured displacements

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Abstract

In this paper we describe a method for determining the elastic constants for paper from a measured displacement field. Our model is based on the finite element method using quadrilateral elements. A method for determining the first and second order derivatives with respect to the unknown parameters plays a crucial role in the optimisation process for solving the inverse problems in this paper. To reduce the influence of the noise of the input data, an efficient continuation method is used for the regularization term in the cost function. Some numerical results for synthetic data and practical measurements will verify the efficiency of our approach.
1. Introduction

Typically the purpose of inverse problems in the context of parameter estimation is to apply an optimisation method to recover a parameter vector $m$ based on observations of a field $u$, where $u$ is related to $m$ by a forward problem, a system of differential equations. The forward problem is often solved by the finite difference or finite element method and the corresponding discrete form can be expressed as

$$f(m, u) = 0, \quad f : \mathbb{R}^p \rightarrow \mathbb{R}^q. \quad (1)$$

Our approach is similar to the one in [1] where a comparison between variants of the Newton method and the Gauss-Newton method was made to show their advantages and disadvantages in recovering the coefficient functions in a system of differential equations. The original purpose in [1] was to recover the conductivity of electromagnetic equation (EM) where the forward problem was solved by the difference method.

In our problem, based on the measured displacements, $u_{meas}$, we want to find $m$ such that

$$\| Q u - u_{meas} \| \leq \| Tol \| \quad (2)$$

or

$$\| u_h(m) - u_{meas} \| \leq Tol, \quad (3)$$

and (1) holds. In [2] $Q$ denotes the projection matrix that maps the finite element solution $u$ at the nodes to the desired measurement locations when the measured points are the nodes of the finite element mesh and $\| Tol \|$ depends on the noise level. The vector $u_h(m)$ is the finite element solution corresponding to the measured points (when they may not be the nodes of the finite element mesh). However, since the data is noisy and the inverse problems of recovering $m$ from (2) (or (3)) and (1) are often ill-posed, a process of regularization is used to recover a relatively smooth (or piecewise smooth) solution to a nearby problem which is unique, at least locally. According to [2] and [3], a model often utilized in practice minimizes the least square residual vector with an added regularization term,

$$\min_{m, u} \quad \frac{1}{2} \| Q u - b \|^2 + \frac{\beta}{2} \| m - m_{ref} \|^2, \quad (4)$$

where $m_{ref}$ is a reference, and $\beta > 0$ is the regularization parameter. The problem (4) is a nonlinear constrained optimisation problem.

In [1], the finite difference method is used to solve the (EM) equation in 1–D, and (1) is then expressed as
\[ A(m)u = q, \]  
(5)

where \( A(m) \) is a square, nonsingular matrix and \( q \) is independent of \( m \). The constrained optimisation problem

\[
\min_{m, u} \frac{1}{2} \left\| Qu - b \right\|^2 + \frac{\beta}{2} \left\| m - m_{\text{ref}} \right\|^2
\]

(6)

can then be written as an unconstrained, nonlinear least squares problem

\[
\min_{m} \frac{1}{2} \left\| QA(m)^{-1} q - b \right\|^2 + \frac{\beta}{2} \left\| m - m_{\text{ref}} \right\|^2.
\]

(7)

However, if the forward problem is solved by the finite element method, the right-handed term \( q \) will be dependent on \( m \) in order to ensure the nonsingularity of the coefficient matrix \( A \). Therefore a modification of the strategy in [1] is needed.

In this paper we try to recover various elastic constants for paper with isotropic and orthotropic description as a plane stress problem. The forward problems based on an equilibrium equation are solved by the finite element method. Both the Gauss-Newton method and the Newton method can be employed to solve the inverse problems. Our emphasis will be on how to obtain the sensitivity matrices and Hessian matrices by performing the first and second order derivatives in detail. To overcome the ill-posedness of the inverse problems caused by the noisy data, a variant of the forward problems and an efficient continuation method will be introduced. Although inspired by the determination of elastic constants in paper, our work is applicable to other materials, which are isotropic or orthotropic.

2. Development of the finite element method

Based on the available practical model for measuring displacements of a rectangular sample paper, i.e., \( \Omega = \{(x, y) \mid 0 \leq x \leq a, 0 \leq y \leq b \} \), where \( a \) and \( b \) are the width and length of the paper, we consider the equilibrium equation

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0, \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0,
\end{align*}
\]

(8)
in which \( \sigma_x \) and \( \sigma_y \) are normal stresses, \( \tau_{xy} \) and \( \tau_{yx} \) are shear stresses, and the body force is neglected. The corresponding boundary conditions can be described as
\[ u = 0, \quad v = 0, \quad \text{for} \quad 0 \leq x \leq a, \quad y = 0, \]
\[ u = 0, \quad v = v(x), \quad \text{for} \quad 0 \leq x \leq a, \quad y = b, \]  \hspace{1cm} (9)
\[ \sigma_x = 0, \quad \tau_{xy} = 0, \quad \text{for} \quad x = 0, \quad 0 \leq y \leq b, \]
\[ \sigma_x = 0, \quad \tau_{xy} = 0, \quad \text{for} \quad x = a, \quad 0 \leq y \leq b. \]

The finite element method is used to solve (8) and (9) implementing quadrilateral elements. The corresponding linear algebraic equation after being assembled is homogeneous, i.e.,

\[ Au = 0 \] \hspace{1cm} (10)

where \( A = A(m) \) and \( m \) is a vector consisting of the elastic constants to be determined. In more detail we have

\[ A = \sum_e K_G \] \hspace{1cm} (11)

where the sum indicates the assembly of the elemental stiffness matrix to the global stiffness matrix and \( K_G \) arises from the expansion of the elemental stiffness matrix \( k_e \) to the global level (see [4]). Indeed, the \( x \) and \( y \) direction components of the displacement at the point \((x, y)\) within the element \( e \) can be represented as

\[ u_e(x, y) = \sum_{i=1}^{4} u_i^e N_i^e(x, y) = N_e^T u_e, \]
\[ v_e(x, y) = \sum_{i=1}^{4} v_i^e N_i^e(x, y) = N_e^T v_e, \] \hspace{1cm} (12)

where \( N_i^e(x, y), i=1,...,4 \) are the element interpolation functions of the quadrilateral element \( e \). Accordingly, the strain can be written as

\[ \varepsilon = B U_e \]

where

\[ U_e^T = \begin{bmatrix} u_1^e & v_1^e & u_2^e & v_2^e & u_3^e & v_3^e & u_4^e & v_4^e \end{bmatrix} \]

is the elemental displacement vector, and \( B \) is referred to as the strain matrix, i.e.,
\[
B = \begin{bmatrix}
\frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\
0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\
\frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x}
\end{bmatrix}
\]  
\tag{13}

In such a way, \( k_e \) can be expressed as

\[
k_e = \int_D B^T C B d\Gamma = t \int_D B^T C B dxdy,
\]  
\tag{14}

where \( t \) is the thickness of the paper and \( C \) is the elasticity matrix for the plane stress problem.

For the isotropic description, \( C = \frac{E}{1-\nu^2} C_{iso} \),

where

\[
C_{iso} = \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1-\nu)/2
\end{bmatrix}
\]  
\tag{15}

\( E \) is the Young modulus and \( \nu \) is the Possion ratio.

For the orthotropic description,

\[
C = \frac{1}{1-\nu_{12}^2 \nu_{21}^2} \begin{bmatrix}
E_1 & E_1 \nu_{21} & 0 \\
E_2 \nu_{12} & E_2 & 0 \\
0 & 0 & (1-\nu_{12}^2 \nu_{21}^2)G_{12}
\end{bmatrix},
\]  
\tag{16}

\( E_1 \) and \( E_2 \) are the Young modulus, \( \nu_{12} \) and \( \nu_{21} \) are the Possion ratio and \( G_{12} \) is the shear modulus.

According to [5], \( C \) can be written as \( C = \frac{E_2}{1-\eta \nu_2^2} C_{oro} \) and
\[
C_{\text{oro}} = \begin{bmatrix}
\eta & \eta v_2 & 0 \\
\eta v_2 & 1 & 0 \\
0 & 0 & \xi(1 - \eta^2)
\end{bmatrix},
\]
where \( \eta = E_1 / E_2, \, v_2 = v_{21}, \, \xi = G_{12} / E_2 \) and the relation \( E_1 v_{21} = E_2 v_{12} \) has been used. The expression (17) is better than (16) when used in the optimisation process.

The fact that (10) is homogeneous implies that the elasticity matrix \( C \) can be replaced by \( C_{\text{iso}} \) and \( C_{\text{oro}} \) for the isotropic and orthotropic description, respectively. Such a simplification of the coefficient matrices of (10) provides no advantage for the calculation of the finite element solutions of the forward problems, but one will observe that it really gives an important simplification in the optimisation procedure. From now on, the elasticity matrix \( C \) is referred to as

\[
C = C_{\text{iso}}
\]
(18)
in the isotropic case and

\[
C = C_{\text{oro}}
\]
(19)
in the orthotropic case unless specified.

Although the general approach to the optimisation method to determine the elastic constants for the isotropic and orthotropic description of paper is the same, we will describe them individually in order to show all the details.

3. Determination of the elastic constants in the orthotropic case

3.1 The optimisation problem

As the optimisation process for the orthotropic description is more complicated than the isotropic description, we first focus on the determination of elastic constants for orthotropic description. By (19), the factor \( E_2 / (1 - \eta^2) \) has been neglected in \( C \) without influencing the solution of (10), i.e., \( C = C_{\text{oro}} \), we only need to determine the reduced parameters \( \eta, v_2, \) and \( \xi \) instead of \( E_2, \eta, v_2 \) and \( \xi \) based on a measured displacement field. To recover the original parameters we use the relation \( E_1 v_{21} = E_2 v_{12} \) that yields \( v_{12} = \eta v_{21} = \eta v_2 \). Once \( \eta, v_{21}, v_{12} \) and \( \xi \) have been determined, the additional information about stress can be used to obtain \( E_2 \). In such a way, all elastic constants can be determined.

Denote the unknown parameters as \( m = [\eta, v_2, \xi]^T \). The measured displacement field is represented as
\[ u_{\text{meas}} = \begin{bmatrix} u_1^\text{meas} & v_1^\text{meas} & u_2^\text{meas} & v_2^\text{meas} & \ldots & u_n^\text{meas} & v_n^\text{meas} \end{bmatrix}^T \]

where the pairs \((u_i^\text{meas}, v_i^\text{meas})\), \(i = 1,2,\ldots,n\), denote the x and y-direction displacement components at the measured locations \((x_i, y_i)\), \(i = 1,2,\ldots,n\). The calculated displacements at the measured locations are gathered in the vector

\[ u^h = \begin{bmatrix} u_1^h & v_1^h & u_2^h & v_2^h & \ldots & u_n^h & v_n^h \end{bmatrix}^T. \quad (20) \]

The pairs \((u_i^h, v_i^h)\), \(i = 1,2,\ldots,n\) denote the x and y-direction displacement components at the measured locations obtained by the finite element method, i.e.,

\[ u_i^h = \sum_{k=1}^{M} u_k N_k(x_i, y_i), \quad (21) \]

\[ v_i^h = \sum_{k=1}^{M} v_k N_k(x_i, y_i), \quad (22) \]

where \(N_k(x,y), k = 1,2,\ldots,M\) are the nodal interpolation functions, \(M\) is the number of the nodes and

\[ u = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & \ldots & u_M & v_M \end{bmatrix}^T \quad (23) \]

satisfies (10). Consequently, \(u_h = u_h(m)\). If all the measured locations belong to the set of mesh nodes of the finite element method, \(u^h\) can be obtained by a projection matrix \(Q\) in the same way as that in [1], i.e., \(u_h = Qu\) and the interpolation denoted by (21) and (22) is not needed.

Our optimisation problem is now

\[ \min_m \frac{1}{2} \left\| u_h(m) - u_{\text{meas}} \right\|^2 + \frac{\beta^2}{2} \left\| m - m_{\text{ref}} \right\|^2. \quad (24) \]

The optimisation problems obtained in this way are typically solved by the Gauss-Newton method or the Newton method. In this section, we will focus on the Gauss Newton method to determine the elastic constants adopting both synthetic and measured data. Furthermore, we will show how to get the second derivatives with respect to the unknown parameters, which are crucial for the Newton method, and the corresponding numerical results.
3.2 Determining the sensitivity matrix

Define the residual as

\[ R(m) = u_R(m) - u_{meas} \]  \hspace{1cm} (25)

and the sensitivity matrix as

\[ J(m) = \nabla R(m) = \begin{bmatrix} \partial r_1 / \partial m_1 & \partial r_1 / \partial m_2 & \ldots & \partial r_1 / \partial m_l \\ \partial r_2 / \partial m_1 & \partial r_2 / \partial m_2 & \ldots & \partial r_2 / \partial m_l \\ \vdots & \vdots & \ddots & \vdots \\ \partial r_n / \partial m_1 & \partial r_n / \partial m_2 & \ldots & \partial r_n / \partial m_l \end{bmatrix} \]  \hspace{1cm} (26)

where \( r_i, i = 1,2,\ldots,n \), is the i-th component of \( R \) and \( l = 3 \) for the orthotropic case. Furthermore, we define

\[ \tilde{R}(m) = [R(m)^T, \beta (m - m_{ref})]^T \]  \hspace{1cm} (27)

and

\[ \tilde{J}(m) = \nabla \tilde{R}(m) = [J^T, \beta I_{3 \times 3}]^T. \]  \hspace{1cm} (28)

The cost function in (24) can be rewritten as

\[ f(m) = \frac{1}{2} \tilde{R}(m)^T \tilde{R}(m) = \frac{1}{2} \|R(m)\|^2 + \frac{\beta^2}{2} \|m - m_{ref}\|^2. \]  \hspace{1cm} (29)

It is well known that the Gauss-Newton method with line search generates a sequence of iteratives, where, for a given current iterative \( m \), an update of the form \( m \leftarrow m + \alpha \delta m \) is subsequently carried out and \( \alpha \) is chosen by a line search. We have used the line search method in [6] with starting value \( \alpha = 1 \). The Gauss-Newton direction \( \delta m \) is obtained by linearizing the expression (29) and then solve

\[ \min_{\delta m} \frac{1}{2} \| J \delta m + \tilde{R}(m) \|^2 \]  \hspace{1cm} (30)

with standard numerical linear algebraic methods.

The calculation of the sensitivity matrix \( J \) is the main challenge. If we can obtain

\[ \nabla u = \begin{bmatrix} \partial u / \partial m_1 & \partial u / \partial m_2 & \partial u / \partial m_3 \end{bmatrix}, \]  \hspace{1cm} (31)
then (26), together with (21), (22) and (25) easily yields $J$. According to the discussion earlier, $u$ satisfies

$$A(m)u = 0.$$  

Because $A(m)$ is the global stiffness matrix corresponding to (8) and (9) before the essential boundary conditions are imposed, it is singular. After imposing the essential boundary conditions, (32) becomes

$$A_0u = q.$$  

Here $A_0 = A_0(m)$ is a symmetric positive definite matrix and the right term $q = q(m)$ is dependent on the unknown parameter vector $m$. If one tries to find the first derivatives based on (33), the process is cumbersome. Instead, we will find the first derivatives appearing in (31) based on (32).

From (32), we get

$$A \frac{\partial u}{\partial m_i} = - \frac{\partial A}{\partial m_i} u$$

and by (11),

$$\frac{\partial A}{\partial m_i} = \sum_{e} \frac{\partial K}{\partial m_i} G, i = 1, 2, 3,$$

which implies that $\partial A / \partial m_i$ can be obtained by assembling $\partial k_e / \partial m_i$ in the same way as $A$. Since $m = [m_1 \quad m_2 \quad m_3]^T = [\eta \quad \nu_2 \quad \xi]^T$, $C$ can be expressed as

$$C = \begin{bmatrix}
m_1 & m_1 m_2 & 0 \\
m_1 m_2 & 1 & 0 \\
0 & 0 & m_3 (1 - m_1 m_2^2)
\end{bmatrix}.$$  

Further,

$$k_e = \int_{D_e} B^T C B d\Gamma$$

and the strain matrix $B$ is independent of $m$ according to (13), it is obvious that

$$\frac{\partial k_e}{\partial m_i} = \int_{D_e} B^T \frac{\partial C}{\partial m_i} B d\Gamma, \quad i = 1, 2, 3,$$
Comparing (14) and (36) shows that the calculation of \( \partial k_e / \partial m_i, \ i = 1, 2, 3 \), only requires the replacement of \( C \) by \( \partial C / \partial m_i, \ i = 1, 2, 3 \) in (14). Thus, we have shown how we can obtain \( \partial A / \partial m_i, \ i = 1, 2, 3 \).

Obviously, (34) and (32) have the same coefficient matrix, i.e., the global stiffness matrix \( A \) before the essential boundary conditions are imposed with different right terms which are known. It is well known that the standard finite element algorithm for solving (8) and (9) yields the process of imposing essential boundary conditions represented in (9) and Cholesky factorisation to solve the subsequent linear algebraic equations with a symmetric positive definite coefficient matrix \( A \).

Similarly, by setting \( \partial u_{(j)} / \partial m_i = 0, \ i = 1, 2, 3 \), if \( u_{(j)} \) corresponds to the essential boundary condition at some boundary node and repeating the same process of imposing essential boundary condition in finite element algorithm in (34), we can obtain three linear algebraic equations with the same coefficient matrix \( A \) and the unknown solution vectors \( \partial u / \partial m_i, \ i = 1, 2, 3 \), respectively. Implementing the Cholesky factorisation of \( A \), \( \partial u / \partial m_i, \ i = 1, 2, 3 \), can be obtained immediately.

Based on the preceding discussion about the sensitivity matrix, we can implement the Gauss-Newton method to obtain the elastic constants.

### 3.3 The Gauss-Newton method (GN) for the orthotropic description

First, in order to check how the noise level of the measurement influences the optimisation process and supply a reference to the admissible accuracy of particular measured displacement, a series of numerical tests using the Gauss-Newton method with line search and the continuation method are applied to some synthetic data generated in the following way.

The sample paper is assumed to be square with length \( a = b = 70 \ mm \) and thickness \( t = 0.1 \ mm \). A typical set of material data for paper with orthotropic description is \( E_1 = 5420, \ E_2 = 1900, \ v_{12} = 0.38, \ v_{21} = 0.14 \) and \( G_{12} = 0.382 \sqrt{E_1 E_2} \), where \( G_{12} \) comes from an estimate called Banm's approximation. Thus we set
\[
m_{acc} = \begin{bmatrix} m_1^{acc} & m_2^{acc} & m_3^{acc} \end{bmatrix}^T = \begin{bmatrix} n^{acc} & v_2^{acc} & \xi^{acc} \end{bmatrix}^T \\
\begin{bmatrix} E_1 & 0 & \frac{G_{12}}{E_2} \\
0 & E_2 & v_{21} \end{bmatrix}^T \approx \begin{bmatrix} 2.85263 & 0.14000 & 0.64519 \end{bmatrix}^T,
\]

which serves as the accurate value. A constant displacement of 0.19 mm in the y direction on the side \(0 \leq x \leq a, y = b\) is assumed, i.e. \(v(x) = 0.19 \text{ mm} \) in (9). Under these conditions, the finite element solution \(U_h\) of the forward problem (8) and (9) corresponding to a \(16 \times 16\) uniform mesh with a certain perturbation is used to generate synthetic displacements at the measured locations in Section 3.3.1 and 3.3.2.

Corresponding to the cost function \(f\) and the components \(m_i\) of the parameter vector \(m\) to be determined, we estimate the typical magnitudes of \(f\) and \(m_i\) and denote them as \(typf\) and \(typm_i\). Then we adopt the stopping criteria described in [6] in our algorithms, i.e.,

\[
\max_{1 \leq i \leq n} \left| \nabla f(m)_i \right| \max \left\{ \left| (m_+)_i \right|, \left| (typm_i)_i \right| \right\} \leq \text{gradtol} \quad (39)
\]

or

\[
\max_{1 \leq i \leq n} \left| (m_+)_i - (m_c)_i \right| \leq \text{steptol}, \quad (40)
\]

where \(m_+\) and \(m_c\) are the solutions to optimisation problem of the latest and current iterations, respectively. Typically \(\text{gradtol} = \text{eps}^{1/3}\) and \(\text{steptol} = \text{eps}^{2/3}\), where \(\text{eps}\) is the roundoff unit.

Actually, our numerical results will show that the Gauss-Newton iteration with line search and the inner iterations within each outer iteration of the continuation method satisfy the stopping criteria (39) or (40) with \(typm = [3.00 \ 0.2 \ 0.6]^T\) and \(typf = 10^{-8}\).

In Algorithm 1, the minimization process with a specific \(\beta\) is referred to as an outer iteration and the Gauss-Newton or Newton iteration with line search within each outer iteration is referred to as an inner iteration.

Algorithm 1

1. Take a relatively large value of \(\beta\), say \(\beta = 0.01\), in (24) for which an almost quadratic problem will be solved.

2. For a fixed \(\beta\), the Gauss-Newton or Newton method with line search is used in the inner iteration taking the solution for the previous \(\beta\) as its initial guess.
(except that the initial guess \( m_0 \) is taken randomly in the first outer iteration) until one of the criteria in (39) and (40) is satisfied or the total number of inner iterations is 20.

3. Update \( \beta \leftarrow \sqrt{0.5\beta} \).

4. If \( \beta > 10^{-5} \), go to step 2.

Since a good starting point generally has been obtained after a few outer iterations, the Gauss-Newton or Newton method without line search can be used directly.

In Sec.3.3.1, the numerical results show the influence of the noise level of the measured displacements and the boundary values on determining the parameter vectors and this can provide a reference for the practical measurement. Then, in Sec.3.3.2, we show the necessity of the continuation method when the noise is larger. Finally, in Sec.3.3.3, we will modify our forward problem based on the measured data and still use the continuation method to obtain the elastic constants for the orthotropic description. The results in Sec.3.3.1 and 3.3.2 are based on the following two cases,

Case 1: \( m_0 = [2.0 \ 0.3 \ 1.2]^T \), \( m_{\text{ref}} = [3.6 \ 0.6 \ 0.8]^T \),

Case 2: \( m_0 = [3.6 \ 0.5 \ 1.0]^T \), \( m_{\text{ref}} = [2.6 \ 0.2 \ 0.8]^T \),

where \( m_{\text{ref}} \) is only used when the regularization parameter \( \beta \neq 0 \) in the cost function.

3.3.1 The influence of the noise level of the measurement and the boundary value

\( U_h \) is the finite element solution based on \( m_{\text{acc}} \) in (38). In this subsection, the synthetic measurement \( u_{\text{meas}} \) is obtained by setting

\[
 u_{\text{meas}}(i) = U_h(i)(1 + r_1v), i = 1, 2, ..., 2n, \quad (41)
\]

where \( r_1 \) is a random number in \([0,1]\), \( v \) denotes the noisy level and \( n \) is the total number of the measured points. For the time being, (41) assumes that each component of \( u_{\text{meas}} \) perturbs in the same relative error. To correspond to the perturbation of the measurement, the boundary value \( v_{\text{meas}}(x) \) on the upper side for the solution \( u_h \) of the forward problem in the optimisation process perturbs in the following way,

\[
 v_{\text{meas}}(x) = v(x)(1 + r_2v), \quad (42)
\]
where \( r = e_r \). Indeed, \( r_1.v \) and \( r.v \) represent the relative perturbation ratio of the synthetic data and the boundary value, respectively, and they become closer and closer when \( e \) tends to 1.

A large number of numerical tests show that the results of the Gauss-Newton method with \( \beta = 0 \) in the cost function in (24) and the continuation method based on GN after the fifteenth outer iteration for different initial guesses accord with each other quite well when the synthetic data are those in (41) and (42), respectively. In fact, the results have at least 4 digital accuracy for them. Therefore, the emphasis in this subsection will be on how the noisy level \( v \) and the ratio \( e = r / r_1 \) influence the accuracy of the determined parameter vector for the orthotropic description and we only list the results obtained by the Gauss-Newton method.

Fix \( r_1 = 0.8 \), Table 1 shows how \( e \) influences the accuracy of the determined parameter vector when \( v = 10^{-2} \). Then set \( r_1 = 0.5 \) and \( v = 10^{-1} \), the corresponding numerical results are shown in Table 2. The results for the initial guesses \( m_0 = [2.0 \ 0.3 \ 1.2]^T \) and \( m_0 = [3.6 \ 0.5 \ 1.0]^T \) have at least six digital accuracy, so we only list the determined parameter \( m \) corresponding the special \( e \) in Table 1 and Table 2.

**Table 1.** The influence of \( e \) on \( m \) when \( v = 10^{-2} \) and \( r = 0.8 \)

<table>
<thead>
<tr>
<th>errorrate</th>
<th>( m^T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.50</td>
<td>[2.14741 0.14254 0.53949]</td>
</tr>
<tr>
<td>1.20</td>
<td>[2.53948 0.14103 0.60001]</td>
</tr>
<tr>
<td>1.10</td>
<td>[2.69046 0.14051 0.62213]</td>
</tr>
<tr>
<td>1.00</td>
<td>[2.85263 0.14000 0.64519]</td>
</tr>
<tr>
<td>0.95</td>
<td>[2.93811 0.13974 0.65709]</td>
</tr>
<tr>
<td>0.90</td>
<td>[3.02659 0.13949 0.66924]</td>
</tr>
<tr>
<td>0.80</td>
<td>[3.21286 0.13897 0.69430]</td>
</tr>
<tr>
<td>0.50</td>
<td>[3.84929 0.13747 0.77537]</td>
</tr>
<tr>
<td>0.00</td>
<td>[5.17454 0.13515 0.92753]</td>
</tr>
</tbody>
</table>

**Table 2.** The influence of \( e \) on \( m \) when \( v = 10^{-2} \) and \( r = 0.5 \)

<table>
<thead>
<tr>
<th>errorrate</th>
<th>( m^T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.20</td>
<td>[1.50594 0.14580 0.42587]</td>
</tr>
<tr>
<td>1.05</td>
<td>[2.39909 0.14154 0.57896]</td>
</tr>
<tr>
<td>1.00</td>
<td>[2.85263 0.14000 0.64519]</td>
</tr>
<tr>
<td>0.98</td>
<td>[3.06285 0.13938 0.67417]</td>
</tr>
<tr>
<td>0.95</td>
<td>[3.41186 0.13847 0.72036]</td>
</tr>
<tr>
<td>0.90</td>
<td>[4.08805 0.13698 0.80425]</td>
</tr>
</tbody>
</table>

From Table 1 and Table 2, the results tend to the accurate value when \( e \) tends to 1. If \( e \) is far from 1, the determined parameter vector \( m \) is terribly inaccurate compared with \( m_{\text{acc}} \). Especially, when \( e = 0 \), \( m \) is completely unacceptable even
\( v = 10^{-2} \). Table 1 shows that, corresponding to the relative perturbing ratio \( r_1, v \) of the synthetic data \( u_{\text{mean}} \) with \( r_1 = 0.8 \) and \( v = 10^{-2} \), the numerical results are close to the accurate value \( m_{\text{acc}} \) in (38) when \( 0.8 \leq e \leq 1.2 \). However, \( e \) has to be in the interval \( (0.95, 1.05) \) to obtain a reasonable approximate solution of \( m_{\text{acc}} \) when \( r_1 = 0.5 \) and \( v = 10^{-1} \). This fact implies that the relative perturbing ratio of the measurement and the boundary value have to be closer to obtain a good approximate solution if the noisy levels of the measurement and the boundary value increase. Motivated by this fact, we have to reduce the noisy level and try to measure the displacement at the measured locations and on the boundary with perturbing ratio as close as possible.

3.3.2 The comparison of the GN with line search and the continuation method

In general, the assumption that the perturbing ratio of each component of the synthetic displacement are the same is not in accordance with the practical measurement. To model the measurement more reasonably, instead of (41) and (42), we set the synthetic data

\[
u_{\text{mean}}(i) = U_y(i),(1 + s(i), v), i = 1, 2, ..., 2n, \tag{43}\]

and the boundary value corresponding to (43)

\[
u_{\text{mean}}(x) = v(x),(1 + r, v), \tag{44}\]

where the components of \( s \) and \( r \) are random numbers in \([0, 1]\). To compare the results, first we fix

\[ r = r_0 \approx 0.65964 \quad \text{and} \quad s = s_0, \]

where the maximum, minimum components and average value of \( s_0 \) approximately equal to 0.99953, 0.00558 and 0.50745, respectively. Table 3 and Table 4 list the determined known vectors \( m^r \) corresponding to the Gauss-Newton method (GN) and the continuation method based on GN (CGN) for Case 1 and Case 2 for \( v = 10^{-2} \) and \( 10^{-1} \), respectively.

Table 3. The results of GN and CGN with \( v = 10^{-2} \)

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>GN</td>
<td>[2.477427 0.141180 0.585333]</td>
</tr>
<tr>
<td>CGN</td>
<td>[2.477453 0.141180 0.585336]</td>
</tr>
</tbody>
</table>

Table 4. The results of GN And CGN with \( v = 10^{-1} \)

<table>
<thead>
<tr>
<th>Case 1</th>
<th>Case 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>GN</td>
<td>[0.806374 0.149952 0.243848]</td>
</tr>
<tr>
<td>CGN</td>
<td>[0.806413 0.149952 0.243855]</td>
</tr>
</tbody>
</table>
From Table 4, we note that the continuation method based on GN can work quite well for two different initial guesses. However, the Gauss-Newton method cannot work at all for some initial guesses. Actually, when the initial guess $m_0 = [3.6 \ 0.5 \ 1.0]$ in Table 4, the computer shows that the positive definiteness of the coefficient matrix of the FEM equation is violated and this implies some $m$ in the iteration is far from the accurate one and may have no physical meaning from a practical viewpoint. Moreover, we have tried many cases in which the initial guess $m_0$, the random vector $s$ and the random number $r$ are different and found that the continuation method based on GN can work very well if $\nu$ is between $10^{-2}$ and $10^{-1}$ from the numerical viewpoint. Nevertheless, the Gauss-Newton method failed for some initial guesses when $\nu$ is larger. Though we can use the simple Gauss-Newton method if the noisy level is lower, the continuation method is very efficient and reliable to determine the parameter vector for the orthotropic description based on the practical measurement whose noisy level may be larger.

As known earlier, the combination of $s$, $r$ and $\nu$ in (43) and (44) determines the noisy level of the synthetic measurement and boundary value. Based on the continuation method, Table 5 will show how these factors influence the results. As the numerical results have at least 4 digital accuracy for Case 1 and Case 2, we only list the results corresponding to Case 1.

Table 5. The relation between $m$ and the noisy level

<table>
<thead>
<tr>
<th>$s_0$</th>
<th>$r_0$</th>
<th>1.0 × 10^{-2}</th>
<th>[2.47745 0.14118 0.58534]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_0$</td>
<td>$r_0$</td>
<td>0.5 × 10^{-2}</td>
<td>[2.65781 0.14059 0.61454]</td>
</tr>
<tr>
<td>$s_0$</td>
<td>$r_0$</td>
<td>0.2 × 10^{-2}</td>
<td>[2.77291 0.14024 0.63276]</td>
</tr>
<tr>
<td>0.4 + 0.1 \times s_0</td>
<td>0.4 + 0.1 \times r_0</td>
<td>1.0 × 10^{-2}</td>
<td>[2.81263 0.14012 0.63897]</td>
</tr>
<tr>
<td>$s_0$</td>
<td>$r_0$</td>
<td>1.0 × 10^{-1}</td>
<td>[0.80641 0.14995 0.24386]</td>
</tr>
<tr>
<td>$s_0$</td>
<td>$r_0$</td>
<td>0.5 × 10^{-1}</td>
<td>[1.45579 0.14567 0.39760]</td>
</tr>
<tr>
<td>$s_0$</td>
<td>$r_0$</td>
<td>0.2 × 10^{-1}</td>
<td>[2.15705 0.14235 0.53109]</td>
</tr>
<tr>
<td>0.4 + 0.1 \times s_0</td>
<td>0.4 + 0.1 \times r_0</td>
<td>1.0 × 10^{-1}</td>
<td>[2.49080 0.14113 0.58753]</td>
</tr>
</tbody>
</table>

Table 5 verifies that the results of the continuation method based on GN are quite satisfactory if the noisy level denoted by $\nu$ is not greater than $10^{-1}$ and the perturbing ratio of the displacement at each measured location and on the boundary are as close as possible. Actually this motivates us to replace the original forward problem (8) and (9) with a variant of it for the practical measurement in Section 3.3.3.

3.3.3 Using measured displacements
As the efficiency of the continuation method has been shown above based on some synthetic data, it is time to fit some practical displacements to obtain the elastic parameters. Except that the fitted displacement is supplied by the measurement, the equilibrium equation, boundary condition and paper sample are still the same as that in Sec.3.3.1 and 3.3.2. Unfortunately, the computation procedure cannot converge even for the continuation method. From the discussion at the end of Sec.3.3.2, perhaps the noise level of the measured displacement is larger and the relative perturbing ratio of the displacement at each measured location and on the boundary are quite different. The improvement of the measurements is helpful, but might be difficult. Instead of an attempt to improve the measurements, we will change the boundary conditions based on the practical measurements. The purpose is to make the perturbing ratio of the displacement at each measured location and on the boundary as close as possible as shown at the end of Section 3.3.2.

The practical measurement of x and y direction displacement is spread over a subdomain \( \Omega_0 \) of the paper sample, \( \Omega = \{(x,y)|0 \leq x \leq 70, 0 \leq y \leq 70\} \), and \( \Omega_0 = \{(x,y) | 2.890625 \leq x \leq 67.109375, 2.890625 \leq y \leq 67.109375\} \). The measured points are uniformly located in \( \Omega_0 \) with the step size of \( x \) and \( y \) direction \( h = 4.28125 \). In such a way, the total measured points are 16 \times 16 = 256. Now we extend \( \Omega_0 \) to \( \Omega_1 = \{(x, y) | 0 \leq x \leq 70, 2.890625 \leq y \leq 67.109375\} = \{(x, y) | 0 \leq x \leq a, b_1 \leq y \leq b_2\} \). Rather than the preceding algorithms devised for (8) and (9) where \( v(x) = 0.19 \text{ mm} \), a constant, we consider the problem

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0, & (x, y) & \in \Omega_1, \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0,
\end{align*}
\]

with boundary conditions

\[
\begin{align*}
u &= v_1(x) & v &= v_2(x) & \text{for} & & 0 \leq x \leq a & y = b_1 \\
u &= u_1(x) & u &= u_2(x) & \text{for} & & 0 \leq x \leq a & y = b_2 \\
\sigma_x &= 0 & \tau_{xy} &= 0 & \text{for} & & x = 0 & b_1 \leq y \leq b_2 \\
\sigma_y &= 0 & \tau_{xy} &= 0 & \text{for} & & x = a & b_1 \leq y \leq b_2
\end{align*}
\]

where \( u_i(x), v_i(x), i = 1,2 \) are the x and y direction component functions of the displacement on the lower and upper boundaries of \( \Omega_1 \) obtained by the measurement applying some interpolation methods. For simplicity, the displacement components at the 4 corner points of \( \Omega_1 \), i.e., \((0, b_2), (0, b_1), (a, b_1) \) and \((a, b_2) \) are obtained by linearly extrapolating the corresponding values at the two neighbouring measured points on the upper and lower boundaries. The point set

\[
\{(0, y_i), (a, y_i) | y_i = b_1 + ih, i = 0,1,...15\}
\]
together with all measured points consists of the mesh points for discretizing $\Omega_1$. Thus the mesh in terms of the finite element calculation for (45) and (46) is $17 \times 15$. All the measured displacement components on upper and lower boundaries of $\Omega_0$, combined with the displacement components at the 4 corner nodes of $\Omega_1$ obtained by linear extrapolation above, serve as the essential boundary conditions.

At this stage, the continuation method will be used to determine the elastic constants for the practical model. Actually, the cost function is simpler than that in (29) as the interpolation procedure (21) and (22) are not needed for the finite element calculation. As a result, the calculation of the sensitive matrix $J$, which is crucial for the optimisation process, is also simpler. We still implement Algorithm 1 in which the Gauss-Newton method with line search serves as the inner iteration to determine the elastic constants for orthotropic description of the practical model. To show that our method is robust, we produce the numerical results related to the following cases. The initial guesses and reference vectors are chosen randomly.

**Case 1:** $m_0 = [2.00 \ 0.30 \ 0.80]^T$, $m_{ref} = [2.00 \ 0.30 \ 0.80]^T$;

**Case 2:** $m_0 = [3.60 \ 0.50 \ 1.00]^T$, $m_{ref} = [3.60 \ 0.50 \ 1.00]^T$;

**Case 3:** $m_0 = [2.85 \ 0.14 \ 0.645]^T$, $m_{ref} = [2.85 \ 0.14 \ 0.645]^T$;

**Case 4:** $m_0 = [2.85 \ 0.14 \ 0.645]^T$, $m_{ref} = [2.00 \ 0.30 \ 0.80]^T$;

**Table 6.** The numerical results when the number of outer iterations is 10

<table>
<thead>
<tr>
<th>m elastic const.</th>
<th>total iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1 $[.67133 \ .39182 \ .29076]^T$, $[.67133 \ .29076 \ .39182 \ .26304]^T$</td>
<td>61</td>
</tr>
<tr>
<td>Case 2 $[.67296 \ .39169 \ .29108]^T$, $[.67296 \ .29108 \ .39169 \ .26359]^T$</td>
<td>69</td>
</tr>
<tr>
<td>Case 3 $[.67217 \ .39175 \ .29091]^T$, $[.67217 \ .29091 \ .39175 \ .26332]^T$</td>
<td>67</td>
</tr>
<tr>
<td>Case 4 $[.67133 \ .39182 \ .29076]^T$, $[.67133 \ .29076 \ .39182 \ .26304]^T$</td>
<td>60</td>
</tr>
</tbody>
</table>

**Table 7.** The numerical results when the number of outer iteration is 15

<table>
<thead>
<tr>
<th>m elastic const.</th>
<th>total iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1 $[.66998 \ .39192 \ .29049]^T$, $[.66998 \ .29049 \ .39192 \ .26258]^T$</td>
<td>70</td>
</tr>
<tr>
<td>Case 2 $[.66999 \ .39192 \ .29049]^T$, $[.66999 \ .29049 \ .39192 \ .26258]^T$</td>
<td>81</td>
</tr>
<tr>
<td>Case 3 $[.66997 \ .39192 \ .29048]^T$, $[.66997 \ .29048 \ .39192 \ .26257]^T$</td>
<td>79</td>
</tr>
<tr>
<td>Case 4 $[.66994 \ .39192 \ .29048]^T$, $[.66994 \ .29048 \ .39192 \ .26257]^T$</td>
<td>69</td>
</tr>
</tbody>
</table>

From Table 6 and Table 7, the iterations of the continuation method converge to the same $m$ though the initial guesses and the reference vectors $m_{ref}$ are quite different from each other. Numerically, this fact verifies that our method not only admits global convergence, but also seems to give a unique solution $m$ to the corresponding optimisation problem. The comparison of Table 4 and Table 5 shows that it is
certainly possible that the Gauss-Newton method without line search replaces the one with line search in the inner iteration after some outer iterations.

Remark 1
First we tried to consider (8) in the measured domain $\Omega_0$ with only the essential boundary conditions constituted by all the measured displacement on $\partial\Omega_0$. Unfortunately, the results are strongly dependent on the choices of initial guesses. Actually they are very close to the initial guesses and certainly unreliable. We conjecture the reason for this is that the residual is always small no matter what the initial guess is. The stopping criteria (39) and (40) in our algorithm cannot be small enough to distinguish the convergent results for different initial guesses.

Remark 2
Secondly, we tried the variant of the forward problem described by (45) and (46) without using the continuation method. We considered two different cases where there is a regularization term or no regularization term. For the first situation, the results are quite different when the regularization term varies. For the second case, the method failed for some initial guesses. As the regularization parameter and the initial guess are not prior, the continuation method is needed for orthotropic description. Nevertheless, we will see the Gauss-Newton method can work very well for the isotropic description of the practical model as shown in Table 12 of Sec.4.1.2.

3.4 The Newton method for the orthotropic description

Besides the Gauss-Newton method, the Newton method is another efficient method to solve optimisation problem. In this subsection, we will use the Newton method to determine the elastic constants for the orthotropic description.

The Newton direction, $\delta m$, is obtained by solving

$$
(\tilde{J}^T \tilde{J} + S) \delta m = -\tilde{J}^T f.
$$

Here $\tilde{J}$ is defined by (28) and

$$
S = \sum_{i=1}^{N} \tilde{r}_i(m) \nabla^2 \tilde{r}_i(m),
$$

where $\tilde{r}_i(m)$ and $N$ are the i-th component and length of residual vector $\tilde{R}(m)$ defined in (27), respectively. The definition of $\tilde{R}(m)$ implies

$$
S = \sum_{i=1}^{n} r_i(m) \nabla^2 r_i(m),
$$
where \( r_i(m) \) and \( n_1 = 2n \) are the \( i \)-th component and length of \( R(m) \) defined in (25), respectively. Except that the Gauss-Newton direction obtained in (30) is replaced by the Newton direction obtained in (47), the contents in this subsection follow the similar strategy in Sec. 3.3. Therefore, the remaining problem is the calculation of \( S \). For this purpose, it is obvious that we only need to calculate 
\[ \frac{\partial^2 R}{\partial m_k \partial m_i}, k,l=1,2,3. \] 
Similar to the calculation of the first derivatives above, the combination of (21), (22), (20) and (23) implies that we only need to obtain 
\[ \frac{\partial^2 u}{\partial m_k \partial m_i}, k,l=1,2,3. \] 
In (32), performing the second derivatives with respect to \( m_k \) and \( m_l \), \( k, l=1, 2, 3 \), we have

\[
A \frac{\partial^2 u}{\partial m_k \partial m_i} = - \frac{\partial^2 A}{\partial m_k \partial m_i} u - \frac{\partial A}{\partial m_k} \frac{\partial u}{\partial m_l} - \frac{\partial A}{\partial m_l} \frac{\partial u}{\partial m_k}, k,l=1,2,3,
\]

where \( u, \frac{\partial u}{\partial m_k} \) and \( \frac{\partial A}{\partial m_k} \), \( k = 1, 2, 3 \) have been derived in Sec. 3.1. If \( \frac{\partial^2 A}{\partial m_k \partial m_i}, k,l=1,2,3 \) are known, \( \frac{\partial^2 u}{\partial m_k \partial m_i}, k,l=1,2,3 \) can be solved by a similar process which was described in detail to obtain \( \frac{\partial u}{\partial m_k} \), \( k=1,2,3 \) in Sec. 3.1. Indeed, by (11),

\[
\frac{\partial^2 A}{\partial m_k \partial m_i} = \sum_{e} \frac{\partial^2 K_e}{\partial m_k \partial m_i}, k,l=1,2,3,
\]

which implies that \( \frac{\partial^2 A}{\partial m_k \partial m_i}, k,l=1,2,3 \) can be obtained by assembling \( \frac{\partial^2 k_e}{\partial m_k \partial m_i}, k,l=1,2,3 \), in the same way as that \( A \) is obtained by assembling \( k_e \). The fact that

\[
k_e = \int_{D_e} B^T C B \, d\Gamma
\]

and the strain matrix \( B \) is independent of \( m \) according to (13) gives that

\[
\frac{\partial^2 k_e}{\partial m_k \partial m_i} = \int_{D_e} B^T \frac{\partial^2 C}{\partial m_k \partial m_i} B \, d\Gamma, k,l=1,2,3,
\]

where

\[
\frac{\partial^2 C}{\partial m_1^2} = 0_{3 \times 3}, \quad \frac{\partial^2 C}{\partial m_2^2} = 0_{3 \times 3},
\]

\[
\frac{\partial^2 C}{\partial m_1 \partial m_2} = \frac{\partial^2 C}{\partial m_2 \partial m_1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2m_2m_3 \end{bmatrix},
\]
The comparison of (14) and (52) shows that for the calculation of
\[ \frac{\partial^2 C}{\partial m_l \partial m_l} = \frac{\partial^2 C}{\partial m_l \partial m_k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -m_l^2 \end{bmatrix}, \tag{55} \]
\[ \frac{\partial^2 C}{\partial m_l \partial m_k} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2m_l m_k \end{bmatrix}, \tag{56} \]
\[ \frac{\partial^2 C}{\partial m_k \partial m_k} = \frac{\partial^2 C}{\partial m_k \partial m_l} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2m_k m_l \end{bmatrix}. \tag{57} \]

The comparison of (14) and (52) shows that for the calculation of
\[ \frac{\partial^2 k}{\partial m_k \partial m_k}, k, l = 1, 2, 3, \]
it is only needed to replace \( C \) by \( \frac{\partial^2 C}{\partial m_k \partial m_k}, k, l = 1, 2, 3 \) in (14). Then \( \frac{\partial^2 A}{\partial m_k \partial m_k}, k, l = 1, 2, 3 \) can be calculated according to (51). At this stage, the Newton method with line search and the continuation method based on the Newton method with line search can be applied to solve the corresponding optimisation problem. After several outer iterations, line search in inner iterations is not needed. As a result, the local quadratic convergence property of the Newton method may be recovered. In the following, we only list two corresponding results obtained by the continuation method based on the GN and the Newton method, respectively, to compare them.

We implement the synthetic data and boundary value defined in (41) and (42) where \( r_i \) is a random number in \([0,1]\) and \( \epsilon = 0 \). The finite element method for the forward problem (8) and (9) is based on a \( 32 \times 32 \) uniform mesh. In Table 8 and Table 9, we list the corresponding results and compare them based on the same initial guess and reference vector, i.e.,

\[ m_0 = [2.0, 0.3, 1.2]^T, \quad m_{ref} = [3.6, 0.6, 0.8]^T. \]

<table>
<thead>
<tr>
<th>outer</th>
<th>( m^l )</th>
<th>iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>[3.87779, 0.13940, 0.81883]</td>
<td>13</td>
</tr>
<tr>
<td>8</td>
<td>[5.25432, 0.13555, 0.95138]</td>
<td>36</td>
</tr>
<tr>
<td>10</td>
<td>[5.53960, 0.13488, 0.97552]</td>
<td>46</td>
</tr>
<tr>
<td>15</td>
<td>[5.66016, 0.13460, 0.98538]</td>
<td>66</td>
</tr>
<tr>
<td>20</td>
<td>[5.66421, 0.13460, 0.98570]</td>
<td>76</td>
</tr>
</tbody>
</table>

Table 8. Numerical results of Algorithm 1 using the GN method for noise level \( 10^{-2} \)
Table 9. Numerical results of Algorithm 1 using the Newton method for noise level $10^{-2}$

<table>
<thead>
<tr>
<th>outer</th>
<th>$m^r$</th>
<th>iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>$[3.87779, 0.13939, 0.81883]$</td>
<td>58</td>
</tr>
<tr>
<td>8</td>
<td>$[5.25431, 0.13555, 0.95138]$</td>
<td>102</td>
</tr>
<tr>
<td>10</td>
<td>$[5.53962, 0.13488, 0.97552]$</td>
<td>109</td>
</tr>
<tr>
<td>15</td>
<td>$[5.66018, 0.13460, 0.98538]$</td>
<td>119</td>
</tr>
<tr>
<td>20</td>
<td>$[5.66425, 0.13460, 0.98571]$</td>
<td>124</td>
</tr>
</tbody>
</table>

where outer and iter represent the outer iterations and the total number of inner iterations, respectively. After 15-20 outer iterations, the continuation method based on the Gauss-Newton or Newton method with line search can work quite well. However, the comparison of Table 8 and Table 9 shows that, at the beginning of the outer iterations, say, the first to the eighth, the method based on Newton converges much more slowly than the one based on GN. However, after the tenth outer iteration, the method based on Newton converges much faster than the one based on GN. Actually the computer time after the tenth outer iteration for Newton is much less than that for GN. However, globally the continuation method based on GN seems to be more efficient and economical than that based on Newton, so we focus on the discussion for GN in this paper.

4. Determination of the elastic constants in the isotropic case

In this section, we will discuss how to determine elastic constants for isotropic description of paper with various methods. The ideas in this section are almost the same as that in Sec. 3, but the process of this section will be shown to be much simpler.

For the isotropic case, the elasticity matrix of the orthotropic case, i.e.,

$$ C = \frac{E_2}{1 - \eta v^2} C_{oro} $$

is replaced by

$$ C = \frac{E}{1 - v^2} C_{iso}, $$

where $C_{oro}$ and $C_{iso}$ are defined in (19) and (18). The equilibrium equation (8), boundary condition (9) and the corresponding finite element method are the same as that for the orthotropic description. Consequently, the factor $E/1 - v^2$ can be neglected in $C$ without changing the solution of (10), i.e., $C = C_{iso}$. As a result, we only need to determine the reduced parameter $v$ instead of $E$, $v$ based on a measured displacement field and this is an optimisation problem with one variable.
In place of the reduced elastic parameters \( m = [\eta \ v_2 \ \xi]^T \) for orthotropic description, \( m = \nu \) for isotropic description. The remaining strategy for the determination of the elastic constant \( \nu \), the expressions of the optimisation problem, the sensitivity matrix and the cost function are almost the same as that in Sec.3.3.

For a given current iterate \( m \), an update of the form \( m \leftarrow m + \alpha \delta m \) is subsequently carried out with \( \alpha \) determined by line search, and the process is repeated to convergence. The correction direction \( \delta m \) is obtained by solving

\[
\min_{\delta m} \frac{1}{2} \| \tilde{J} \delta m + \tilde{R} \|_2^2
\]

for the Gauss-Newton method or

\[
(\tilde{J}^T \tilde{J} + S) \delta m = -\tilde{J}^T f
\]

for the Newton method, where

\[
S = \sum_{i=1}^{n_1} r_i(m) \nabla^2 r_i(m)
\]

and \( n_1 = 2n \) is the length of \( R(m) \). Recalling the expressions (25), (21), (22) and (20), we only need to calculate \( \partial u / \partial m \) and \( \partial^2 u / \partial m^2 \) to obtain \( \tilde{J} \) and \( S \) for isotropic description. As \( C = C_{iso} \), we deduce

\[
\frac{\partial C}{\partial m} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1/2 \end{bmatrix}
\]

and

\[
\frac{\partial^2 C}{\partial m^2} = 0_{3 \times 3}.
\]

Further \( \partial u / \partial m \) and \( \partial^2 u / \partial m^2 \) can be calculated based on (10) and the details are almost the same as those in Sec.3.2 and 3.4. For simplicity, we omit them here.

At this stage, we can perform various algorithms above and compare them with each other.

### 4.1 Numerical experiments corresponding to the isotropic case

Except that we try to determine the Possion ratio \( \nu \) and view \( \nu_{acc} = m_{acc} = 0.25 \) as the accurate parameter, the sample paper, the forward problem, the boundary
conditions and the method of producing the synthetic data for the isotropic case are the same as those for the orthotropic case in Sec.3.3.

By means of the comparison of various methods, the advantage of our method in Sec.4.1.2 will be shown. The stopping criterion (39) and (40) with \( \text{typm} = 0.2 \) and \( \text{typf} = 10^{-8} \) is still used in the Gauss-Newton iteration with line search and the inner iterations of the continuation method.

### 4.1.1 Numerical results based on the synthetic data for the isotropic case

To model the practical measurement, only the synthetic data (43) and (44) are adopted with \( s = s_0 \) and \( r = r_0 \). We take the initial guess in the admissible interval \((0,1)\) for the Possion ratio randomly and find that both of the GN with line search and the continuation method for isotropic description whose detail is the same as that in Algorithm 1 can work quite well for \( v = 10^{-2}, 10^{-1} \) and \( 10^{0} \) respectively. Indeed, the results of the continuation method converge after 1-2 outer iteration and are in accordance with the results of the GN very well. Therefore, in terms of several typical initial guesses, i.e., \( m_0 = 0.2, 0.4, 0.8 \), only the GN with line search is used in this subsection. Based on different initial guesses, the numerical results have at least 7 digital accuracy for \( v = 10^{-2}, 10^{-1} \) and \( 10^{0} \) respectively, so we only list the relation of \( v \) and the determined vector \( m \) in Table 10.

**Table 10.** Numerical results of the isotropic material based on the synthetic data

<table>
<thead>
<tr>
<th>( v )</th>
<th>( 10^{-2} )</th>
<th>( 10^{-1} )</th>
<th>( 10^{0} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m )</td>
<td>0.24941</td>
<td>0.24447</td>
<td>0.21496</td>
</tr>
</tbody>
</table>

From Table 10, the GN with line search can determine the Possion ratio quite accurately even the order of perturbation is \( 10^{-1} \). This, together with the fact that \( s = s_0 \) and \( r = r_0 \) in the synthetic data, implies that the influence of the order of the noise level and the perturbing ratio of each component of the synthetic displacement and the boundary value is much smaller for isotropic material than that for orthotropic material. Accordingly, the optimisation procedure without a regularization term can be used to determine the Possion ratio \( v \) if the relative perturbation of the displacement at the measured locations and on the boundary is not more than \( 10^{-1} \).

### 4.1.2 The determination of parameters of the isotropic description for the practical model

Now we return to the practical measurements described in Sec.3.3.3. At first the synthetic data in Sec.4.1.1 is replaced by the measurements. In this case, the algorithm cannot work well even if the continuation method is used. We guess the cause for this is that the measurement of the displacement and the boundary value is too bad. For improving this situation, we have implemented the model (45) and (46) described in Sec.3.3. and the continuation method. The numerical results corresponding to three different cases are listed in Table 11.
Case 1: $m_0 = 0.4, \ m_{\text{ref}} = 0.2$
Case 2: $m_0 = 0.6, \ m_{\text{ref}} = 0.2$
Case 3: $m_0 = 0.9, \ m_{\text{ref}} = 0.2$

Table 11. The numerical results of isotropic description for the practical model based on the continuation method

<table>
<thead>
<tr>
<th>outer</th>
<th>1</th>
<th>3</th>
<th>4</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>0.37396340849817</td>
<td>0.37400447409818</td>
<td>0.37401132609045</td>
<td>0.37401817510374</td>
</tr>
<tr>
<td>Case 2</td>
<td>0.37396340850663</td>
<td>0.37400447409818</td>
<td>0.37401132609045</td>
<td>0.37401817416315</td>
</tr>
<tr>
<td>Case 3</td>
<td>0.37396340833023</td>
<td>0.37400447409817</td>
<td>0.37401132609045</td>
<td>0.37401817676963</td>
</tr>
</tbody>
</table>

where outer is the outer iterations. Comparing the results of the fourth outer iteration with the regularization parameter $\beta = 3.5 \times 10^{-3}$ and the 20-th outer iteration with $\beta = 1.381 \times 10^{-5}$, we find the results differ little. In fact, after the first outer iteration, the numerical results are very stable as only one inner iteration is needed in each outer iteration.

Figur 1.

Motivated by these results above, we try to simplify the calculation by only adopting the preceding variant of the forward problem without using the continuation method, i.e., there is no regularization term in the cost function (29) and the Gauss-Newton method with line search is implemented. The numerical results corresponding to the three cases in Table 11 are listed in Table 12. As there is no regularization term in the cost function, $m_{\text{ref}}$ is not needed.
Table 12. The practical model without regularization term

<table>
<thead>
<tr>
<th>$m_0$</th>
<th>$m$</th>
<th>iter</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4</td>
<td>0.37401817863631</td>
<td>3</td>
</tr>
<tr>
<td>0.6</td>
<td>0.37401817864444</td>
<td>4</td>
</tr>
<tr>
<td>0.9</td>
<td>0.37401817848989</td>
<td>5</td>
</tr>
</tbody>
</table>

From Table 12, it seems that there exists a unique minimum point for the cost function without regularization term in the admissible range $v \in (0,1)$. To check this, we plot the corresponding figure of the cost function versus $v$ in Figure 1.

Table 11 and Table 12 show that the regularization term and then the continuation method are not needed for the determination of the parameters of isotropic description for the practical model. This fact is opposite to the case for orthotropic description as shown in Remark 2 in Sec.3.3.3. Therefore, the orthotropic description is much more difficult to handle than the isotropic description.

Similar to the discussion for the orthotropic description, the additional information about stresses can be used to obtain $E$.

5. The main codes related to the determination of elastic constants

test70.mat supplies the measured displacements and the coordinators coming from the practical measurements.

test70.mat
- Purpose: Supply the messages about the coordinators and displacements at the measured points denoted by $x_{mat}$, $y_{mat}$, $u$ and $v$ respectively which have to be loaded in all the codes below. The paper sample is square with length $a = b = 70mm$. The boundary conditions are described by (9) with $v(x) = 0.19mm$ when the synthetic data plays the role of the measurement.

5.1 The codes for the orthotropic description

All the codes in this subsection are used to determine the 3-D vector $m$ and the 4-D vector $elastic$ corresponding to orthotropic description. We first list the main variables:

- $x_0$: The initial guess consisting of 3 components
- $solution$, $solution1$: The parameter vectors at convergence or the end of the inner iterations in each outer iteration.
- $elastic$, $elastic1$: The elastic constants at convergence or the end of the inner iterations in each outer iteration.
- $nx$, $ny$: The intervals of $x$ and $y$ direction in the FEM mesh.
- **variance**: The order of the noise level for the synthetis data.
- **beta**: Regularization parameter.
- **xref**: The reference parameter.
- **measdifftol**: the tolerance of $|| uh - u_{fit} || / || u_{fit} ||$.

**gsfixtest.m**

- **Purpose**: Implementing the Gauss-Newton method with line search without regularization term to determine the elastic constants for the synthetic data

$$u_{\text{meas}} = U_h + \text{rand}(\text{length}(U_h),1) \times \text{variance}$$

and boundary value fixed.

- **Input**: $x_0(row) \in R^3$, $nx$, $ny$ and **variance**.

- **Output**: solution, elastic.

- **Conclusion**: The order of the noise level cannot be larger than $10^{-4}$ when only the Gauss-Newton method with line search without any regularization term is used. Even the order of the noise level is small, the initial guess cannot be too far from the accurate one. However, in experimental measurement of the displacement, the noise level of data is usually much greater than $10^{-4}$. So this method has to be improved.

- Corresponding to gsfixtest.m, there are several variants of it except that the synthetic data are different. We list their file name and the corresponding synthetic data in the following.

  a) **gsfixtestper2.m**: The synthetic data

  $$u_{\text{meas}} = U_h \times (1 + \text{ram var 1}. \text{variance})$$

  and

  $$v_{\text{meas}}(x) = v(x) \times (1 + \text{ram var}. \text{variance})$$

  with error rate $\text{ram var}/\text{ram var 1}$ varied in the interval $[0,2]$ and $\text{ram var 1} = 0.8$.

  b) **gsfixtestper3.m**: The synthetic data

  $$u_{\text{meas}} = U_h \times (1 + \text{ram var 1}. \text{variance})$$

  and

  $$v_{\text{meas}}(x) = v(x) \times (1 + \text{ram var}. \text{variance})$$

  with error rate $\text{ram var}/\text{ram var 1}$ varied in the interval $[0,2]$ and $\text{ram var 1} = 0.5$.

  c) **gsfixtestperimvar.m**: The synthetic data

  $$u_{\text{meas}}(i) = U_h(i) \times (1 + \text{synrand}(i). \text{variance}), \ i = 1,2,\ldots,2n,$$
and the boundary value

$$v_{\text{meas}}(x) = v(x)(1 + \text{ram var \ variance}),$$

where the components of $\text{synrand}$ and $\text{ram var}$ are random numbers in the interval $[0,1]$ produced by computer randomly.

**gnfixre3comp.m**
- **Purpose:** Implementing the Continuation method based on GN to determine the elastic constants for the synthetic data

$$u_{\text{meas}} = U_h(1 + \text{rand \ variance})$$
and boundary value fixed.

- **Input:** initial guess $x_0 \in R^3$ (column), $nx, ny$, initial regularization parameter $\beta$ (suggested value 0.01), variance, $xref$(column) and measdiff tol.

- **Output:** the outer iteration $outer$, the inner iterations $itcount$, $solution1$ and $elastic1$ corresponding to each outer iteration.

- **Conclusion:** The initial guesses can be quite different from the accurate value and the accuracy of the numerical results is at least 4 digits for completely different initial and reference models even if the noise level is $10^{-2}$. However, if the noise level reaches $10^{-1}$, the numerical results are not so good. This fact shows that the noise level has to be limited to $10^{-2}$ for the continuation method based on GN for orthotropic description.

- **Corresponding to gnfixre3comp.m, there are several variants of it except that the synthetic data are different. We list them and the corresponding synthetic data in the following.**

  d) **gnfixre3comper3.m:** The synthetic data

  $$u_{\text{meas}} = U_h(1 + \text{ram var \ variance})$$
  and

  $$v_{\text{meas}}(x) = v(x)(1 + \text{ram var \ variance})$$

  with $\text{errorrate} = \text{ram var} / \text{ram var1}$ varied in the interval $[0,2]$ and $\text{ram var1} = 0.8$.

  e) **gnfixre3comper4.m:** The synthetic data

  $$u_{\text{meas}} = U_h(1 + \text{ram var \ variance})$$
  and

  $$v_{\text{meas}}(x) = v(x)(1 + \text{ram var \ variance})$$
with  \( \text{error rate} = \text{ram} \text{ var} / \text{ram} \text{ var 1} \) varied in the interval \([0,2]\) and \( \text{ram} \text{ var 1} = 0.5 \).

f) \text{gnfixre3comperimvar.m}: The synthetic data

\[
\text{u}_{\text{meas}}(i) = U_h(i)(1 + \text{synrand}(i) \cdot \text{variance}), \ i = 1, 2, ..., 2n,
\]

and the boundary value

\[
\nu_{\text{meas}}(x) = \nu(x)(1 + \text{ram} \text{ var} \cdot \text{variance}),
\]

where the components of \( \text{synrand} \) and \( \text{ram} \text{ var} \) are random numbers in the interval \([0,1]\) produced by computer randomly.

\text{nfixre3comp.m}

- Purpose: Implementing the Continuation method based on the Newton method to determine the elastic constants for the synthetic data

\[
\text{u}_{\text{meas}} = U_h(1 + \text{rand} \cdot \text{variance})
\]

and boundary value fixed.

- Input: initial guess \( x_0 \in R^3 \) (column), \( nx, ny \), initial regularization parameter \( \beta \) (suggested value 0.01), \( \text{variance}, x_{\text{ref}} \) (column) and \( \text{measdift tol} \).

- Output: the outer iteration \( \text{outer} \), the corresponding regularization parameter \( \beta \); the inner iterations \( \text{itcount}, \text{solution1} \) and \( \text{elastic1} \) corresponding to each outer iteration.

- Conclusion: At the beginning of the outer iterations the method based on FN converges much more slowly than the one based on GN. However, after the tenth outer iteration, the method based on FN converges much faster than the one based on GN. Actually the computer time after the tenth outer iteration for FN is much less than that for GN. Globally the continuation method based on GN seems to be more efficient and economic than that based on FN.

\text{testmar2homo.m}

- Purpose: Implementing the Continuation method based on GN to determine the elastic constants for practical model. A variant of the forward problem is introduced.

- Input: initial guess \( x_0 \in R^3 \) (row), \( nx = 17, ny = 15 \) according to the messages about the measured point, initial regularization parameter \( \beta \) (suggested value 0.01), \( x_{\text{ref}} \) (column) and \( \text{measdift tol} \).

- Output: the outer iteration \( \text{outer} \), the corresponding regularization parameter \( \beta \); the inner iterations \( \text{itcount}, \text{solution1} \) and \( \text{elastic1} \) corresponding to each outer iteration.
Conclusion: The combination of the continuation method and the variant of the forward problem is robust. The iterations almost converge to the same $x$ though the initial guesses and the references $x_{\text{ref}}$ are quite different from each other. Actually our numerical tests showed both the continuation method and the variant of the forward problem are needed, otherwise one cannot obtain any reasonable results.

5.2 The codes for the isotropic description

All the codes in this Subsection are used to determine the Possion ratio $v = x$ for isotropic description. Except that the unknowns and reference parameters are 1-D instead of 3-D, the remaining variables are almost the same as that in sec.5.1.

oneisoresidual.m

- Purpose: Implementing the Gauss-Newton method with line search to determine the Possion ratio $v$ and check the sensitivity of initial guesses to the order of the noise level comparing that in orthotropic description ($gsfixtest.m$ above) when there is no regularization term added. Here the synthetic data are

$$u_{\text{meas}} = U_h \cdot (1 + \text{rand}.\text{variance})$$

and the boundary value fixed.

- Input: initial guess $x_0 \in R$, $nx ny$, the regularization parameter $\beta = 0$, $\text{variance}$, $x_{\text{ref}}$ (as $\beta = 0$, $x_{\text{ref}}$ has no use in this calculation).

- Output: the approximation of the Possion ratio $\text{solution}$ and the total iterations $\text{itcountin}n$.

- Conclusion: The results are quite satisfactory if the order of noise level can be limited by $10^{-2}$ and the influence of the order of the noise level is much smaller for isotropic material than that for orthotropic material. Accordingly, the optimisation procedure without regularization term and cooling process can be used to determine the Possion ratio $v$ if the measurement is relatively accurate. However the initial guess has to be chosen appropriately.

Oneisognim.m: A variant of oneisoresidual.m except that the synthetic data

$$u_{\text{meas}}(i) = U_h(i) \cdot (1 + \text{synrand}(i).\text{variance}), i = 1, 2, ..., 2n,$$

and the boundary value

$$v_{\text{meas}}(x) = v(x) \cdot (1 + \text{ram var}.\text{variance})$$

where the components of $\text{synrand}$ and $\text{ram var}$ are random numbers in the interval $[0,1]$ produced by computer randomly.
gnisocool.m

- **Purpose:** To obtain the global convergence by using the continuation method in the same way as that in `gnfixre3comp.m` for orthotropic description. Here the synthetic data are

\[ u_{\text{meas}} = U_h(1 + \text{rand} \cdot \text{variance}) \]

and the boundary value fixed.

- **Input:** initial guess \( x_0 \in R \), \( nx, ny \), the regularization parameter \( \beta \) (suggested value 0.01), \( \text{variance} \), \( x_{\text{ref}} \in R \) and \( \text{measdiff tol} \).

- **Output:** the outer iteration \( \text{outer} \), the corresponding regularization parameter \( \beta \); the inner iterations \( \text{itcount} \) and \( \text{solution1} \) (representing \( v \)) corresponding to each outer iteration.

- **Conclusion:** The numerical results show that the continuation method is not necessary when the noise level is not more than \( 10^{-1} \) for isotropic material compared with the fact that the noise level has to be limited not more than \( 10^{-2} \) and the continuation method plays a crucial role for orthotropic material.

Oneisocgnim: A variant of `gnisocool.m` except that the synthetic data are

\[ u_{\text{meas}}(i) = U_h(i)(1 + \text{synrand}(i) \cdot \text{variance}), i = 1, 2, ..., 2n, \]

and the boundary value

\[ v_{\text{meas}}(x) = v(x)(1 + \text{ram var} \cdot \text{variance}), \]

where the components of \( \text{synrand} \) and \( \text{ram var} \) are random numbers in the interval \((0, 1)\) produced by computer randomly.

testmarisohomo.m

- **Purpose:** Implementing the Continuation method based on GN to determine the Possion ratio \( v \) of isotropic description for practical model. A variant of the forward problem is introduced.

- **Input:** initial guess \( x_0 \in R \), \( nx = 17, ny = 15 \) according to the messages about the measured points, initial regularization parameter \( \beta \) (suggested value 0.01), \( x_{\text{ref}} \in R \) and \( \text{measdiff tol} \).

- **Output:** the outer iteration \( \text{outer} \), the corresponding regularization parameter \( \beta \); the inner iterations \( \text{itcount} \), \( \text{solution1} \) (representing \( v \)) corresponding to each outer iteration.

- **Conclusion:** The numerical results are very stable as only one inner iteration is needed in each outer iteration after the first outer iteration and the iterative
solutions are almost the same in each outer iteration. This suggests that the continuation method is not needed, but the variant of the forward problem is crucial. This case is opposite to that for orthotropic description in testmar2homo.m.

**testmarnore.m**

- Purpose: To check the efficiency of the process in which only the variant of the forward problem is used without any continuation method and regularization term. Still the Gauss-Newton method is implemented.

  - Input: initial guess $x_0 \in R, nx, ny$, the regularization parameter $beta = 0$, $xref \in R$. ($xref$ has no use here as $\beta = 0$).

  - Output: solution (representing $v$), and the total iteration $itcountinn$.

- Conclusion: The regularization term and then the continuation method are not needed for the determination of the parameters of isotropic description for the practical model. This fact is opposite to the case for the orthotropic description as shown in testmar2homo.m. However, the variant of the forward problem is crucial for the practical model.

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7. **References**


