G-Convergence and Homogenization of some Monotone Operators

Marianne Olsson

DEPARTMENT OF
ENGINEERING, PHYSICS AND MATHEMATICS

Mid Sweden University Doctoral Thesis 45
G-CONVERGENCE AND HOMOGENIZATION OF SOME MONOTONE OPERATORS

Marianne Olsson

Supervisors:
Associate Professor Anders Holmbom, Mid Sweden University
Professor Nils Svanstedt, Göteborg University
Professor Mårten Gulliksson, Mid Sweden University

Department of Engineering, Physics and Mathematics
Mid Sweden University, SE-831 25 Östersund, Sweden

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Marianne Olsson

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Department of Engineering, Physics and Mathematics
Mid Sweden University, SE-831 25 Östersund
Sweden

Telephone: +46 (0)771-97 50 00

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Marianne Olsson
Department of Engineering, Physics and Mathematics
Mid Sweden University, SE-831 25 Östersund, Sweden

Abstract

In this thesis we investigate some partial differential equations with respect to $G$-convergence and homogenization. We study a few monotone parabolic equations that contain periodic oscillations on several scales, and also some linear elliptic and parabolic problems where there are no periodicity assumptions. To begin with, we examine parabolic equations with multiple scales regarding the existence and uniqueness of the solution, in view of the properties of some monotone operators. We then consider $G$-convergence for elliptic and parabolic operators and recall some results that guarantee the existence of a well-posed limit problem. Then we proceed with some classical homogenization techniques that allow an explicit characterization of the limit operator in periodic cases. In this context, we prove $G$-convergence and homogenization results for a monotone parabolic problem with oscillations on two scales in the space variable. Then we consider two-scale convergence and the homogenization method based on this notion, and also its generalization to multiple scales. This is further extended to the case that allows oscillations in space as well as in time. We prove homogenization results for a monotone parabolic problem with oscillations on two spatial scales and one temporal scale, and for a linear parabolic problem where oscillations occur on one scale in space and two scales in time. Finally, we study some linear elliptic and parabolic problems where no periodicity assumptions are made and where the coefficients are created by certain integral operators. Here we prove results concerning when the $G$-limit may be obtained immediately and is equal to a certain weak limit of the sequence of coefficients.
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Östersund, January 2008
Marianne Olsson
Notations

For the convenience of the reader, here are some symbols and sets used in what follows. For some fundamental definitions and results from functional analysis, the reader is referred to the appendix.

\(X\) : Any linear space.
\(\|u\|_X\) : The norm of \(u \in X\), when \(X\) is a normed space.
\(X'\) : The dual space of \(X\).
\(H\) : Any Hilbert space.
\(V\) : Any Banach space such that the embedding \(V \subseteq H\) is continuous.
\(\{u^h\}\) : A sequence of functions \(u^h\).
\(u^h \to u\) : \(\{u^h\}\) converges strongly to \(u\).
\(u^h \rightharpoonup u\) : \(\{u^h\}\) converges weakly to \(u\).
\(u^h \rightharpoonup^* u\) : \(\{u^h\}\) converges weakly* to \(u\).
\(\{\varepsilon\}\) : A sequence \(\{\varepsilon(h)\}\) such that \(\varepsilon = \varepsilon(h) \to 0\) as \(h \to \infty\).
\(\mathcal{O}\) : Any open bounded subset of \(\mathbb{R}^M\) with smooth (at least Lipschitz) boundary.
\(\partial\mathcal{O}\) : The boundary of \(\mathcal{O}\).
\(\bar{\mathcal{O}}\) : The closure of \(\mathcal{O}\).
\(\Omega\) : Any open bounded subset of \(\mathbb{R}^N\) with smooth (at least Lipschitz) boundary.
\(\Omega_T\) : The set \(\Omega \times (0,T)\).
\(\bar{\Omega}_T\) : The set \(\bar{\Omega} \times [0,T]\).
\(Y\) : Unit cell in \(\mathbb{R}^M\).
\(Y, Y_1, Y_2, \ldots Y_n\) : Unit cells in \(\mathbb{R}^N\).
\(Y^n\) : The set \(Y_1 \times \cdots \times Y_n\).
\(\mathcal{V}_{n,m}\) : The set \(Y^n \times (0,1)^m\).
\(a \cdot b\) : The scalar product of two vectors \(a\) and \(b\) in \(\mathbb{R}^N\).
\((u,v)_H\) : The inner product of \(u\) and \(v\) in a Hilbert space \(H\).
\(\langle \cdot, \cdot \rangle_{X',X}\) : The duality pairing between \(X'\) and \(X\).
Below is a list of function spaces and their norms. All functions $u$ are assumed to be measurable.

$F (\mathcal{O})$: Any space of functions $u : \mathcal{O} \rightarrow \mathbb{R}$.

$F_{\text{loc}}(\mathbb{R}^M)$: All functions $u : \mathbb{R}^M \rightarrow \mathbb{R}$ such that their restriction to any open bounded subset $\mathcal{O}$ of $\mathbb{R}^M$ belongs to $F (\mathcal{O})$.

$F_{\text{p}}(Y_*)$: All functions in $F_{\text{loc}}(\mathbb{R}^M)$ which are the periodic repetition of some function in $F (Y_*)$.

$F (\mathcal{O})/\mathbb{R}$: All functions $u$ in $F (\mathcal{O})$ such that $\int_{\mathcal{O}} u (x) \, dx = 0$.

$L^p (\mathcal{O})$: All functions $u : \mathcal{O} \rightarrow \mathbb{R}$ such that
$$\|u\|_{L^p(\mathcal{O})} = \left( \int_{\mathcal{O}} \left| u (x) \right|^p \, dx \right)^{1/p} < \infty.$$  

$L^\infty (\mathcal{O})$: All functions $u : \mathcal{O} \rightarrow \mathbb{R}$ such that
$$\|u\|_{L^\infty(\mathcal{O})} = \text{ess sup}_{x \in \mathcal{O}} \left| u (x) \right| < \infty.$$  

$W^{1,p} (\mathcal{O})$: All functions $u$ in $L^p (\mathcal{O})$ such that their first-order distributional derivatives belong to $L^p (\mathcal{O})$.
$$\|u\|_{W^{1,p}(\mathcal{O})} = \|u\|_{L^p(\mathcal{O})} + \|\nabla u\|_{L^p(\mathcal{O})^M}.$$  

$W^{1,p}_0 (\mathcal{O})$: All functions $u$ in $W^{1,p} (\mathcal{O})$ such that $u = 0$ on $\partial \mathcal{O}$.
$$\|u\|_{W^{1,p}_0(\mathcal{O})} = \|\nabla u\|_{L^p(\mathcal{O})^M}.$$  

$W^{-1,q} (\mathcal{O})$: The dual space of $W^{1,p}_0 (\mathcal{O})$, $\frac{1}{p} + \frac{1}{q} = 1$.

$H^1 (\mathcal{O})$: The function space $W^{1,p} (\mathcal{O})$ for $p = 2$.

$H^1_0 (\mathcal{O})$: The function space $W^{1,p}_0 (\mathcal{O})$ for $p = 2$.

$H^{-1} (\mathcal{O})$: The space $W^{-1,q} (\mathcal{O})$ for $p = q = 2$.

$C(\bar{\mathcal{O}})$: All continuous functions $u : \bar{\mathcal{O}} \rightarrow \mathbb{R}$.
$$\|u\|_{C(\bar{\mathcal{O}})} = \sup_{x \in \bar{\mathcal{O}}} |u (x)|.$$  

$C_0(\mathcal{O})$: All continuous functions $u : \mathcal{O} \rightarrow \mathbb{R}$ with compact support in $\mathcal{O}$.

$C^\infty (\mathcal{O})$: All infinitely differentiable functions $u : \mathcal{O} \rightarrow \mathbb{R}$.  

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$D(\mathcal{O})$: All infinitely differentiable functions $u : \mathcal{O} \to \mathbb{R}$ with compact support in $\mathcal{O}$.

$D'(\mathcal{O})$: All distributions in $\mathcal{O}$.

$C^*_c(Y_*)$: All continuous and $Y_*$-periodic functions $u : \mathbb{R}^M \to \mathbb{R}$.

\[\|u\|_{C^*_c(Y_*)} = \sup_{y \in Y_*} |u(y)|\]

$C^\infty_c(Y_*)$: All infinitely differentiable and $Y_*$-periodic functions $u : \mathbb{R}^M \to \mathbb{R}$.

$H^1_0(Y_*)$: All $Y_*$-periodic functions in $H^1_{loc}(\mathbb{R}^M)$.

$H^1_0(Y_*)/\mathbb{R}$: All functions $u$ in $H^1_0(Y_*)$ such that $\int_{Y_*} u(y) \, dy = 0$.

\[\|u\|_{H^1_0(Y_*)/\mathbb{R}} = \|\nabla u\|_{L^2(Y_*)^M}\]

$L^\infty_0(Y_*)$: All $Y_*$-periodic functions in $L^\infty(\mathbb{R}^M)$.

$L^2(\mathcal{O}; X)$: All functions $u : \mathcal{O} \to X$ such that

\[\|u\|_{L^2(\mathcal{O}; X)} = (\int_{\mathcal{O}} \|u(x, \cdot)\|_X^2 \, dx)^{1/2} < \infty.\]

$L^2(0,1; X)$: All $(0,1)$-periodic functions $u : \mathbb{R} \to X$ such that

\[\|u\|_{L^2(0,1; X)} = (\int_0^1 \|u(x, \cdot)\|_X^2 \, dx)^{1/2} < \infty.\]

$L^\infty(\mathcal{O}; X)$: All functions $u : \mathcal{O} \to X$ such that

\[\|u\|_{L^\infty(\mathcal{O}; X)} = \text{ess sup}_{x \in \mathcal{O}} \|u(x)\|_X < \infty.\]

$C(\bar{\mathcal{O}}; X)$: All continuous functions $u : \bar{\mathcal{O}} \to X$.

\[\|u\|_{C(\bar{\mathcal{O}}; X)} = \sup_{x \in \mathcal{O}} \|u(x, \cdot)\|_X\]

$C_0(\mathcal{O}, X)$: All continuous functions $u : \mathcal{O} \to X$ with compact support in $\mathcal{O}$.

$D(\mathcal{O}; X)$: All infinitely differentiable functions $u : \mathcal{O} \to X$ with compact support in $\mathcal{O}$.

$H^1(0,T; V,V')$: All functions $u$ in $L^2(0,T; V)$ such that $\partial_t u$ belongs to $L^2(0,T; V')$.

\[\|u\|_{H^1(0,T; V,V')} = \|u\|_{L^2(0,T; V)} + \|\partial_t u\|_{L^2(0,T; V')}\]
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1 Introduction

In fields such as physics and engineering, there is often cause to study physical phenomena in highly heterogeneous media. Typical problems are heat conduction, elasticity or torsion in composites and flow in porous media, just to mention a few. Composites consist of two or more different materials that are mixed together in e.g. a fibered, layered or crystalline structure and are widely used because of their properties which may combine beneficial features of the individual components. Some well-known composites are reinforced concrete, plastic reinforced by glass or carbon fiber and engineered wood such as plywood. Heterogeneous materials with a fine microstructure also occur naturally, such as for example in wood, porous rocks and bone.

![Figure 1. A material with a fine microstructure.](image)

The microscopic structure makes problems of the type mentioned above difficult to treat directly. Typically, they can be modeled by some partial differential equation (PDE), where the presence of the microscopic variations is reflected in oscillations in the coefficient of the equation (or in a perforated domain in the case of porous media). Solving such a problem using some numerical method would require a very fine mesh to capture the changes in the structure. Thus, the solution may be costly to find or be out of reach.

When the microstructure is very fine compared to the sample of material, as in Figure 1, the material may at first glance seem to be homogeneous, and we can also expect it to behave as if homogeneous at the macroscopic level. This means that another approach to these problems might be to search for this global behavior; that is, to look for the macroscopic or so-called effective properties of the material. Thus, using the information about the microstructure, such as the properties of the constituents and the arrangement of these, the aim is to "homogenize" the material in the sense of finding the properties of a homogeneous material that gives the same overall response as the heterogeneous one.
1.1 The idea of homogenization

The main source of inspiration for the mathematical theory of homogenization is to find the effective properties in problems such as those mentioned above. A prototype problem is stationary heat diffusion in a composite material with a periodic microstructure. We briefly describe some features of homogenization illustrated by this problem.

We let the set $\Omega$ be occupied by our composite material. If we assume that the heterogeneities are evenly distributed, we can model the material as periodic, as in Figure 2.

As illustrated in the figure, this means that we can think of the material as being built up of small identical cubes, the side length of which we call $\varepsilon$. We let $a(y)$ be the periodic repetition of the function that describes how the ability to conduct heat varies over the representative cell $Y$; see Figure 2. For simplicity, we choose $Y$ to be the unit cube. Hence, substituting $\xi$ for $y$, we obtain a function $a(\xi)$ that oscillates periodically with period $\varepsilon$ as the variable $x$ passes through $\Omega$, describing the oscillations of the heat conductivity in the composite.

If the material sample is located in an environment with temperature zero and is subject to a heat source given by a function $f$, after a while the temperature distribution in the material will stabilize to the solution $u^\varepsilon$ to the problem

\[-\nabla \cdot \left( a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon (x) \right) = f(x) \quad \text{in} \quad \Omega, \tag{1}\]

\[u^\varepsilon (x) = 0 \quad \text{on} \quad \partial \Omega.\]

This is the usual stationary heat equation, governed by $a \left( \frac{\xi}{\varepsilon} \right)$, the heat conductivity of our periodic material.

Figure 2. The material sample and the representative cell.
For small values of $\varepsilon$, equation (1) will be very difficult to treat numerically, which means that we might not be able to compute $u^{\varepsilon}$. On the other hand, when $\varepsilon$ is small we can expect that the material would behave like a homogenous one and hence, from a macroscopic point of view, could be described by an equation of the form

$$-\nabla \cdot (b \nabla u(x)) = f(x) \quad \text{in } \Omega,$$

$$u(x) = 0 \quad \text{on } \partial \Omega,$$

with $b$ constant. Since this problem does not contain any rapid oscillations, it is much easier to solve than the original one. Thus, if $b$ was known, we could find $u$, which would give us a good approximation of the temperature distribution.

So how could we find the effective property represented by $b$? For the periodic case a first, and not unreasonable, guess might be that $b$ is given by the arithmetic mean value of $a$ over a period, i.e. over $Y$. We try this for a one-dimensional example with $\Omega = (0, 1)$, $f(x) = x^2$ and

$$a(y) = \frac{1}{2 + \sin 2\pi y}$$

and solve

$$-\frac{d}{dx} \left( \int_Y a(y) \, dy \frac{d}{dx} u(x) \right) = f(x) \quad \text{in } \Omega,$$

$$u(x) = 0 \quad \text{on } \partial \Omega.$$

As we can see from Figure 3 below, the solution $u$ does not give a good approximation of the true temperature distribution $u^{\varepsilon}$, which means that our guess was wrong.

![Figure 3](image-url)
To find $b$, the thermal conductivity of the fictitious homogeneous material, we imagine that the structure becomes increasingly finer, as in Figure 4.

![Figure 4. Homogenizing the mixture.](image)

This corresponds to the study of a sequence of equations (1) for successively smaller values of $\varepsilon$. If the sequence $\{u^\varepsilon\}$ of solutions close in on some function $u$ that does not have any rapid oscillations depending on $\varepsilon$, this function should give the desired approximation of the temperature distribution. This is illustrated in Figure 5 for example (3).

![Figure 5. $u^\varepsilon$ for some values of $\varepsilon$ and the limit $u$.](image)

Thus, if by letting $\varepsilon$ approach zero in (1), we could find a problem of the form (2), which provides us with the limit function $u$ as its unique solution, we would obtain the proper macroscopic description of the original problem. In the following sections we will study some mathematical aspects of this homogenization problem in more detail; that is, what type of convergence we can expect for $\{u^\varepsilon\}$, and how we can pass to the limit and find the so-called homogenized problem.

### 1.2 The snag

To find the limit equation, we want to let $\varepsilon$ pass to zero in (1). To do this, we will apply certain types of weak convergence, and the proper form of the equation is then the so-called weak form. If we let $f \in L^2(\Omega)$, this means that we are searching for $u^\varepsilon \in H^1_0(\Omega)$ such that

$$
\int_{\Omega} a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx
$$

for all $v \in H^1_0(\Omega)$.
Under standard assumptions on boundedness and coercivity of $a$, see Section 3.1.1, this problem has a unique solution $u^\varepsilon$, also called the weak solution to (1). From the properties of $a$ we also get that $\{u^\varepsilon\}$ satisfies a so-called a priori estimate; that is, boundedness in a certain function space, in this case $H^1_0(\Omega)$. Since $H^1_0(\Omega)$ is a reflexive Banach space, this guarantees that there is a function $u$ such that, at least for a subsequence, it holds that

$$ u^\varepsilon (x) \rightharpoonup u (x) \quad \text{in} \quad H^1_0(\Omega). \quad (6) $$

Next, we want to pass to the limit in (5) to find the problem for which this limit $u$ is the unique solution. As it turns out, this is a rather intricate task. Through the assumptions made and the separability of $L^1(\Omega)$ we have

$$ a \left( \frac{x}{\varepsilon} \right) \rightharpoonup \frac{1}{y} \int_Y a (y) \, dy \quad \text{in} \quad L^\infty (\Omega)^{N \times N}, $$

which implies weak convergence in $L^2(\Omega)^{N \times N}$ to the same limit. Furthermore, the a priori estimate gives that $\{\nabla u^\varepsilon (x)\}$ is bounded in $L^2(\Omega)^N$ and hence it holds, up to a subsequence, that

$$ \nabla u^\varepsilon (x) \rightharpoonup \nabla u (x) \quad \text{in} \quad L^2(\Omega)^N, $$

where $u$ is the limit in (6). These convergence properties mean that if only one of the sequences $\{a \left( \frac{x}{\varepsilon} \right)\}$ and $\{\nabla u^\varepsilon (x)\}$ was present in (5), we could immediately pass to the limit, at least for a subsequence, just by replacing the sequence with its weak limit. However, when taking the product, as in (5), it is not that easy. Generally, the product of two only weakly convergent sequences does not converge to the product of the limits. Even though the sequence $\{a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon (x)\}$ of products is bounded in $L^2(\Omega)^N$, see Section 3.1.1, in general we have

$$ a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon (x) \not\rightharpoonup \int_Y a (y) \, dy \, \nabla u (x) \quad \text{in} \quad L^2(\Omega)^N, $$

which means that usually

$$ b \neq \int_Y a (y) \, dy. $$

This is the explanation as to why our initial guess was not very good after all. We remark that in one dimension $b$ turns out to be the harmonic mean of $a$, but in higher dimensions this does generally not hold true.
Thus, the homogenized coefficient $b$ is usually not any conventional type of mean of $a$. Nor is it in general any usual kind of weak limit of $\{a(\frac{x}{e})\}$, and obviously there is no trivial way of passing to the limit in (5) and hence finding the homogenized problem. Motivated by the identification of this major snag in the homogenization process, we now turn our interest to some of the tools developed for proving the existence of, and finding, the limit $b$.

### 1.3 $G$-convergence

The matter of the existence and the character of the limit coefficient can be embedded in the general theory of $G$-convergence, which is a type of convergence of operators, first introduced by Spagnolo in [Spa1] and [Spa2] for second-order elliptic and parabolic operators. It can be applied for example to problems of the form

$$\begin{align*}
-\nabla \cdot (a^h(x) \nabla u^h(x)) &= f(x) \quad \text{in } \Omega, \\
u^h(x) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

where the coefficient matrices $a^h$ do not have to meet any periodicity requirements. In its simplest form, $G$-convergence of $\{a^h\}$ means that the sequence $\{u^h\}$ of solutions to (7) satisfies

$$u^h(x) \rightharpoonup u(x) \quad \text{in } H^1_0(\Omega)$$

where $u$ solves an equation of the type

$$\begin{align*}
-\nabla \cdot (b(x) \nabla u(x)) &= f(x) \quad \text{in } \Omega, \\
u(x) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

and $b$, which we call the $G$-limit of $\{a^h\}$, has appropriate properties so that (8) has a unique solution. This type of convergence will be investigated in Chapter 3.

If $\{a^h\}$ is a sequence of symmetric matrix functions that fulfill certain conditions of uniform boundedness and coercivity, we will have $G$-convergence up to a subsequence, and there is also a similar result for the case without symmetry assumption. This means that at least for a subsequence, we are guaranteed the existence of a limit problem with suitable properties. This is the case for the model problem, for example. However, we do not know what $b$ looks like and we have seen that it is not trivial to determine $b$ even in the periodic case.
1.4 Periodic homogenization

Much effort has been made to develop methods for finding the limit matrix $b$, especially for periodic problems. We will mention some of the most important ones here, but we first take a look at their common result for the prototype problem (1).

It turns out that the information needed to characterize $b$ in (2) can be attained from a so-called local problem, which is an equation defined on one representative cell $Y$, in our case the unit cube. In the homogenized problem the coefficient has the entries

$$b_{ij} = \int_Y a_{ij}(y) + \sum_{k=1}^N a_{ik}(y) \partial y_k z_j(y) \, dy.$$  \hfill (9)

Each $z_j$ can, under certain assumptions of periodicity, be determined from a corresponding local problem

$$-\nabla_y \cdot (a(y) (e_j + \nabla_y z_j(y))) = 0 \quad \text{in} \ Y,$$

where $\{e_j\}_{j=1}^N$ is the usual orthonormal basis in $\mathbb{R}^N$, and hence we can identify the limit coefficient $b$.

In (9) there is an additional term compared to the kind of mean that we used in (4). The fact that $b$ does not coincide with this mean is also obvious from Figure 6, where we show these numbers together with $a(\frac{x}{\varepsilon})$ for example (3).

![Figure 6](image)

Figure 6. The coefficients $a(\frac{x}{\varepsilon})$ and $b$, and the arithmetic mean of $a$.

Once $b$ has been identified, we can easily compute the solution $u$ to the homogenized problem (2). Now $u$ agrees well with the solution $u^\varepsilon$, as illus-
trated in Figure 7 for $\varepsilon = 0.05$.

![Figure 7. The solutions $u^\varepsilon$ and $u$.](image)

Since the limit problem is created in such a way that it is solved by the weak $H^1_0(\Omega)$-limit, and hence the strong $L^2(\Omega)$-limit, of $\{u^\varepsilon\}$, the correspondence between the functions will improve for smaller values of $\varepsilon$. Obviously, the second term in (9) has captured the effect of the oscillations in $a(\frac{x}{\varepsilon})$.

Thus, homogenization provides us with the $G$-limit, representing the effective property discussed in Section 1.1. Also, the numerical computation is considerably simpler than solving the original equation (1) for $\varepsilon$ close to zero, in the sense that there are no rapid oscillations in the equations we have to solve, i.e. the local and the homogenized problem.

A classical method for derivation of the local and the homogenized problem, widely used in mechanics and physics, is the asymptotic expansion method; see e.g. [BLP], [SaPa] and [CiDo]. With this approach, one assumes that the solution $u^\varepsilon$ admits a two-scale asymptotic expansion of the form

$$u^\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \ldots,$$

(10)

where each $u_k$ is $Y$-periodic in the second argument. Plugging (10) into equation (1) and equating equal powers of $\varepsilon$, we formally obtain both the local problem and the homogenized problem. The main features of this method are given in Chapter 4.

To achieve this result rigorously, one passes to the limit in (5)—making appropriate choices of test functions $v$. One method from classical homogenization theory, which uses this approach, is the method of oscillating test functions, originally introduced by Tartar; see e.g. [Tar1] and [CiDo]. This technique is also sketched in Chapter 4.
Another method, a generalization of which will be used to prove some of the main results in this thesis, is the method based on so-called two-scale convergence. This type of convergence was first presented by Nguetseng in [Ngu1]. If \( \{ u^\varepsilon \} \) is bounded in \( L^2(\Omega) \), it holds, up to a subsequence, that for some \( u_0 \) in \( L^2(\Omega \times Y) \)

\[
\int_\Omega u^\varepsilon(x) v(\frac{x}{\varepsilon}) \, dx \to \int_\Omega \int_Y u_0(x,y) v(x,y) \, dy \, dx
\]

for all \( v \) in \( L^2(\Omega \times Y) \) that are sufficiently smooth and periodic in the second variable. The most significant property of this type of convergence is that the limit \( u_0 \) contains both the global and the local variable. This makes the method well suited for problems with periodically oscillating coefficients. We will return to this homogenization technique and its generalizations in Chapter 5.

1.5 Outline of the thesis

In this thesis, we will study a few different equations with respect to \( G \)-convergence and homogenization. Some of the main contributions concern periodic parabolic problems with several micro-scales, i.e., problems of the form

\[
\partial_t u^\varepsilon(x,t) - \nabla \cdot a^\varepsilon(x,t, \nabla u^\varepsilon) = f(x,t) \quad \text{in } \Omega_T,
\]

\[
u^\varepsilon(x,0) = u^0(x) \quad \text{in } \Omega,
\]

\[
u^\varepsilon(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T),
\]

for some choices of \( a^\varepsilon \) with periodic oscillations that depend on \( \varepsilon \) in various ways. We also investigate some elliptic and parabolic problems without periodicity assumptions.

The thesis is organized as follows. In Chapter 2 we discuss the properties of some monotone operators and parabolic PDEs. In particular, we will see under which assumptions we will have a unique solution to equation (11) for our choices of \( a^\varepsilon \).

In Chapter 3 we define \( G \)-convergence for elliptic and parabolic operators and recall some results on the existence of the \( G \)-limit. In connection with this, we also comment on some properties of these limits. Finally, we discuss methods to obtain the \( G \)-limit for parabolic problems from the corresponding results for elliptic equations.
The most well-studied case of $G$-convergence is periodic homogenization. In Chapter 4, we become acquainted with a couple of classical homogenization techniques, i.e. the method of asymptotic expansions and Tartar’s method of oscillating test functions. These allow an explicit characterization of the $G$-limit in periodic problems and we illustrate how they can be applied to homogenize the model problem (1). We then proceed with problems with periodic oscillations on several microscopic scales. Adopting the terminology of [BLP], the homogenization of problems containing multiple scales will be referred to as reiterated homogenization. After recalling a result for monotone elliptic problems with two microscopic spatial scales, we study the corresponding parabolic case, that is, problem (11) with

$$a^\varepsilon (x, t, \nabla u^\varepsilon) = a \left( t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u^\varepsilon \right).$$

We prove the existence of a $G$-limit for this problem and also the corresponding homogenization result, using the result for elliptic problems and a comparison theorem for $G$-limits.

Chapter 5 is devoted to the more recent homogenization method based on the concept of two-scale convergence and its generalizations. First, we give a brief description of two-scale convergence and demonstrate how it can be applied for the homogenization of the model problem. We also investigate how a mode of convergence of two-scale type can contribute to the interpretation of the asymptotic expansion. We then study the extension of two-scale convergence to the multiscale case and illustrate reiterated homogenization for an elliptic problem with two spatial microscopic scales. Furthermore, we introduce evolution multiscale convergence, which allows oscillations also in time, and study the homogenization of two parabolic problems where such oscillations occur. First, we make the preparations necessary to perform the homogenization of these problems. In particular, we prove a compactness theorem for sequences of gradients under certain boundedness assumptions. We then consider a problem where there, in addition to the two spatial microscopic scales, is one fast temporal scale; that is, we have chosen

$$a^\varepsilon (x, t, \nabla u^\varepsilon) = a \left( \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right).$$

We carry out the homogenization procedure for this problem, where we identify local problems and characterize the limit function $b$ for $0 < r < 3$. Here
we distinguish three different cases: $0 < r < 2$, $r = 2$ and $2 < r < 3$, for which we have different local problems. We also study a linear parabolic problem which contains one spatial scale at the microscopic level and where we have introduced a second fast temporal scale, i.e. (11) with

$$a^{\varepsilon}(x, t, \nabla u^{\varepsilon}) = a\left(\frac{x}{\varepsilon^r}, \frac{t}{\varepsilon^r}\right) \nabla u^{\varepsilon}(x, t).$$ \hspace{1cm} (14)

We end this chapter by homogenizing this problem for $r > 0$, $r \neq 1$, where we consider the cases $0 < r < 2$ with $r \neq 1$, $r = 2$ and $r > 2$.

Finally, in Chapter 6 we study $G$-convergence for certain problems where there are no periodicity assumptions. We investigate some linear elliptic and parabolic problems with coefficients created by means of operators of the type

$$a^{h}_{ij}(x) = \int_{A} w^{h}_{ij}(x, y) K_{ij}(x, y) \, dy$$

and

$$a^{h}_{ij}(x, t) = \int_{A} w^{h}_{ij}(x, t, y) K_{ij}(x, t, y) \, dy,$$

respectively, where $A$ is an open bounded set in $\mathbb{R}^M$ and $\{w^{h}\}$ converges weakly in suitable Lebesgue spaces. Here, we will see under which circumstances we can immediately conclude that the $G$-limit is equal to the weak limit of $\{a^{h}\}$ in $L^2(\Omega)$ and $L^2(\Omega_T)$, respectively.

Most results in this thesis can also be found in papers [FHOSv], [FlOl2], [FlOl3] or [HOS], which have been published in international journals, or in the contribution [FHOSi1] to the proceedings of an international conference. The investigation in Chapter 2 of existence and uniqueness of solutions to parabolic PDEs with multiple scales is a slightly more general version of a result found in [FlOl2]. The result in Chapter 4 on $G$-convergence for case (12) is proven along the lines of the proof of a theorem in [FlOl2], whereas the homogenization result for the same problem is found in [FHOSv]. Most results in Chapter 5 can be found in [FlOl2] and [FlOl3]. Case (13) is treated in [FlOl2], where both the existence of the $G$-limit and the characterization of this limit by means of homogenization are investigated, and in [FlOl3] the homogenization result for case (14) is proven. Chapter 6 is based on [HOS] and [FHOSi1] and extensions of the results found there.
2 Monotone operators

Some of the PDEs we will investigate in this thesis contain operators which may depend on $\nabla u$ in a nonlinear way. In this chapter we review some theory of monotone operators which will allow us to study such problems, especially regarding what we need to require to be assured of the existence of a unique solution.

In Section 2.1 we start out with an elementary case of a real equation containing a monotone function, where we know the conditions sufficient for a unique solution to exist. We also consider the generalization of this result to stationary monotone operator equations, which can be applied e.g. to elliptic PDEs. Then we proceed with the case in which the monotone operator appears in an evolution problem in Section 2.2. Finally, we apply the result on the existence of a unique solution for general evolution problems to study the parabolic PDE (11) for the monotone cases (12) and (13).

2.1 The concept of monotone operators

Let us start with an example from elementary calculus. Consider

$$A(u) = f,$$

where $u$ and $f$ are in $\mathbb{R}$, and we assume that $A$ fulfills the following conditions:

(I*) $A : \mathbb{R} \to \mathbb{R}$ is strictly monotone.

(II*) $A$ is continuous.

(III*) $A(u) \to \pm \infty$ as $u \to \pm \infty$.

From (II*) and (III*) it follows, by the intermediate value theorem, that this equation has a solution for any $f$ in $\mathbb{R}$. Moreover, since $A$ is strictly monotone the solution is unique.

This case, which is quite easily understood, can be generalized to problems of the type

$$Au = f,$$  \hspace{1cm} (15)

where $u$ is in $X$, which may now be any Banach space, and $f$ is in the dual space $X'$. To be precise, since this equation is an equality between functionals
in $X'$, acting on elements in $X$, (15) means that
\[
\langle Au, v \rangle_{X',X} = \langle f, v \rangle_{X',X}
\]
for all $v \in X$. In other words, for $u \in X$, the corresponding operator $Au \in X'$ should have the same effect on all $v \in X$ as the functional $f$.

Monotonicity of the operator $A : X \to X'$ means that
\[
\langle Au - Av, u - v \rangle_{X',X} \geq 0
\]
for all $u, v \in X$. This is of course consistent with the real case. Indeed, for $X = X' = \mathbb{R}$, (17) simply expresses that $A(u) - A(v)$ and $u - v$ always have the same sign, and hence $A$ is increasing.

As in the real case, we need the stronger assumption of strict monotonicity to obtain uniqueness of the solution to (15). This is given in (I) below for operators on general Banach spaces. We also give conditions $(II^\ast)$ and $(III^\ast)$ in generalized form.

(I) The operator $A : X \to X'$ is strictly monotone on $X$, i.e.,
\[
\langle Au - Av, u - v \rangle_{X',X} > 0
\]
for all $u, v \in X$.

(II) $A$ is hemicontinuous, i.e., the function
\[
g(\delta) = \langle A(u + \delta v), w \rangle_{X',X}
\]
is continuous on $[0, 1]$ for all fixed $u, v, w \in X$.

(III) $A$ is coercive, i.e.,
\[
\lim_{\|u\|_X \to \infty} \frac{\langle Au, u \rangle_{X',X}}{\|u\|_X} = \infty.
\]

These conditions will guarantee the existence and uniqueness of the solution to (15) as stated in the next theorem.

**Theorem 1** If $A : X \to X'$ satisfies the conditions $(I)$-$(III)$ and $X$ is a real reflexive Banach space, then (15) possesses a unique solution $u \in X$.

**Proof.** See, e.g., Theorem 26.A in [ZeiIIB], or [Min] and [Bro]. ■
Let us also mention the concept of maximal monotone operators, which we will refer to in Section 4.2. An operator \( A : X \to X' \) is called \textit{maximal monotone} if for any \((u, f) \in X \times X'\) such that
\[
\langle f - Av, u - v \rangle_{X',X} \geq 0
\]
for every \( v \) in \( X \), it follows that \( f = Au \); see e.g. A.3.2. in [Def]. This means that if \( A \) is maximal monotone, we have an alternative way of expressing the equation \( Au = f \).

The following proposition tells us that one way of knowing that an operator is maximal monotone is to show that it is monotone and hemicontinuous. A proof can be found, e.g., in [Lio].

**Proposition 2** Let \( X \) be a Banach space and \( A : X \to X' \) a monotone and hemicontinuous operator. Then \( A \) is maximal monotone.

The theory of monotone operators can be applied, e.g., to elliptic PDEs of the type
\[
-\nabla \cdot a(x, \nabla u) = f(x) \quad \text{in } \Omega,
\]
\[
u (x) = 0 \quad \text{on } \partial \Omega.
\]

More precisely, we study the weak form of (19) which states that for \( f \in H^{-1}(\Omega) \), we search for \( u \) in \( H^1_0(\Omega) \) such that
\[
\int_\Omega a(x, \nabla u) \cdot \nabla v(x) \, dx = \langle f, v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}
\]
holds for all \( v \in H^1_0(\Omega) \). This means that the PDE can be interpreted as an equality between functionals. Let us consider the operator
\[
A : H^1_0(\Omega) \to H^{-1}(\Omega)
\]
given by
\[
\langle A(\cdot), \cdot \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \int_\Omega a(x, \nabla(\cdot)) \cdot \nabla (\cdot) \, dx.
\]
Following the pattern above for the general case, we obtain for each \( u \in H^1_0(\Omega) \) a functional \( Au \in H^{-1}(\Omega) \) defined by
\[
\langle Au, \cdot \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \int_\Omega a(x, \nabla u) \cdot \nabla (\cdot) \, dx
\]
and hence equation (20) is of the type (16).
If we impose certain requirements on $a$ the operator $A$ fulfills the sufficient conditions (I)-(III) given above, and hence there is a unique $u \in H^1_0(\Omega)$ such that (20) holds for all $v \in H^1_0(\Omega)$. The following conditions on the Lebesgue measurable function

$$a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N,$$

where $\alpha, \beta > 0$, $0 < k_1 \leq 1$ and $2 \leq k_2 < \infty$ are enough to ensure this; see Lemma 1 in [LNW].

(i) $a(x, 0) = 0$ for a.e. $x \in \Omega$.

(ii) $|a(x, \xi) - a(x, \xi')| \leq \beta (1 + |\xi| + |\xi'|)^{1-k_1} |\xi - \xi'|^{k_1}$
for a.e. $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$.

(iii) $(a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') \geq \alpha (1 + |\xi| + |\xi'|)^{2-k_2} |\xi - \xi'|^{k_2}$
for a.e. $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$.

Remark 3 A solution $u \in H^1_0(\Omega)$ to (20) is called a weak solution to (19). In the following, all solutions to PDEs are solutions in this sense.

Remark 4 In the discussion above we have considered the Hilbert space setting of elliptic PDEs; that is, for right-hand sides in $H^{-1}(\Omega)$ and solutions in $H^1_0(\Omega)$. The corresponding can be done for equations where we consider source terms in $W^{-1,q}(\Omega)$ and the solution is searched for in $W^{1,p}_0(\Omega)$ for $1 < p < \infty$; see, e.g., [LNW] and [ZeiIIB]. Let us also remark that the conditions (i)-(iii) above are sometimes restricted to special cases, which simplifies the form of the conditions. This will be done in Chapter 3, where we fix $k_2 = 2$ and define $G$-convergence for the corresponding functions.

2.2 Monotone operators and evolution problems

In order to study equation (11), which is a time-dependent problem, we want to extend the results in the previous section to evolution problems. We start with a general evolution problem. The theory for the general case is then carried over to equation (11). In particular, we will state conditions that are enough to ensure the existence of a unique solution.

In later chapters, we will see that these conditions are also sufficient to obtain certain a priori estimates and, at least for a subsequence, to have $G$-convergence of the operators corresponding to (11). Except for being a prerequisite for $G$-convergence, the a priori estimates are necessary to perform the homogenization procedures in Section 5.3.
2.2.1 General evolution problems

Let us consider the problem
\[ \partial_t u(t) + A(t)u(t) = f(t) \quad \text{for a.e. } t \in (0, T), \]
\[ u(0) = u^0, \]
for \( f \in L^2(0, T; V') \) and \( u^0 \in H \), where \( H \) is a Hilbert space containing the Banach space \( V \) and \( A(t) : V \to V' \) creates operators \( A(t)u \in V' \) for each \( t \in (0, T) \) and any \( u \in V \). Our aim is to identify conditions on \( A \) that provide (21) with a unique solution
\[ u \in H^1(0, T; V, V') = \{ u \in L^2(0, T; V) | \partial_t u \in L^2(0, T; V') \}. \]

The two spaces \( V \) and \( H \) that appear in this evolution problem are assumed to fulfill the conditions of a so-called evolution triple, which can be described as follows. Consider a real, separable and reflexive Banach space \( V \) and a real, separable Hilbert space \( H \) such that \( V \) is continuously embedded in \( H \); that is, \( V \subseteq H \) and
\[ \|u\|_H \leq C \|u\|_V \]
for all \( u \in V \). Furthermore, let \( V \) be dense in \( H \). We then say that
\[ V \subseteq H \subseteq V' \]
forms an evolution triple.

Note also that the derivative \( \partial_t u \) should be understood as a generalized derivative; see Section 23.5 in [ZeiIIA]. This will also be the interpretation throughout the thesis.

Again, the problem (21) should be interpreted as that we are searching for \( u \in H^1(0, T; V, V') \) such that for almost every \( t \in (0, T) \), the equation
\[ \langle \partial_t u, v \rangle_{V', V} + \langle A(t)u, v \rangle_{V', V} = \langle f(t), v \rangle_{V', V}, \]
\[ u(0) = u^0 \]
holds for all \( v \in V \).

We assume that the operator \( A \) has the following properties, where the first three essentially correspond to the conditions listed for the stationary case, and the two additional ones concern the time-dependence.
The operator $A(t) : V \to V'$ is **monotone**, i.e.

$$\langle A(t)u - A(t)v, u - v \rangle_{V', V} \geq 0$$

(23)

for all $u, v \in V$ and any $t \in (0, T)$.

$A(t)$ is **hemicontinuous**; that is, the function

$$g_1(\delta) = \langle A(t)(u + \delta v), w \rangle_{V', V}$$

is continuous on $[0, 1]$ for all fixed $u, v, w \in V$ and any $t \in (0, T)$.

$A(t)$ is **coercive**; that is, it satisfies

$$\langle A(t)u, u \rangle_{V', V} \geq C_1 \|u\|^2_V$$

(24)

for all $u \in V$, some $C_1 > 0$ and any $t \in (0, T)$.

$A(t)$ satisfies the **growth condition** that there exists a non-negative function $g_2 \in L^2(0, T)$ and a constant $C_2 > 0$ such that

$$\|A(t)u\|_{V'} \leq g_2(t) + C_2 \|u\|_V$$

for all $u \in V$ and any $t \in (0, T)$.

The function

$$g_3(t) = \langle A(t)u, v \rangle_{V', V}$$

is **measurable** on $(0, T)$ for all fixed $u, v \in V$.

**Theorem 5** Consider the initial boundary value problem (21) where $V \subseteq H \subseteq V'$ forms an evolution triple. If $A$ satisfies the conditions (I)–(V), this problem possesses a unique solution $u \in H^1(0, T; V, V')$.

**Proof.** See Theorem 30.A. in [ZeiIIB].  

**Remark 6** Condition (III) above is somewhat stronger than in Theorem 30.A. in [ZeiIIB], to be in better accordance with our investigations in the remainder of the thesis. Let us also mention that the coercivity condition of the form (18) used in the stationary case follows from (24) if we divide by the norm of $u$ in the latter form. Note also that this evolution setting does not require strict monotonicity; see Section 30.3a. in [ZeiIIB].
### 2.2.2 Parabolic PDEs with multiple scales

The theory for general evolution problems from the preceding section can be applied to the study of equations of the character

\[
\partial_t u(x, t) - \nabla \cdot a(x, t, \nabla u) = f(x, t) \quad \text{in } \Omega_T, \\
u(x, 0) = u^0(x) \quad \text{in } \Omega, \\
u(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T),
\]

i.e. parabolic PDEs, which is a special case of the equation given in (21). To study a problem of this type, we may choose \( V \) to be \( H^{1,0}_0(\Omega) \), which is a real, separable and reflexive Banach space. Also, we can choose \( H \) to be equal to \( L^2(\Omega) \) since this space fulfills the properties required. It is a real, separable Hilbert space and, moreover, it holds that \( H^{1,0}_0(\Omega) \) is dense in \( L^2(\Omega) \) and, by the Poincaré inequality,

\[
\|u\|_{L^2(\Omega)} \leq C \|u\|_{H^{1,0}_0(\Omega)}
\]

for all \( u \in H^{1,0}_0(\Omega) \). Hence, we can form the evolution triple

\( H^{1,0}_0(\Omega) \subseteq L^2(\Omega) \subseteq H^{-1}(\Omega) \).

As in the stationary case, it is possible to specify conditions for \( a \) such that (25) has a unique solution for any \( f \in L^2(0, T; H^{-1}(\Omega)) \) and any \( u^0 \in L^2(\Omega) \).

In this thesis we will investigate a couple of different equations of the type (25), which are governed by operators containing rapid oscillations in space or in both time and space, i.e. problem (11) for the cases (12) and (13). These problems are covered by the equation

\[
\partial_t u^\varepsilon(x, t) - \nabla \cdot a\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \nabla u^\varepsilon\right) = f(x, t) \quad \text{in } \Omega_T, \\
u^\varepsilon(x, 0) = u^0(x) \quad \text{in } \Omega, \\
u^\varepsilon(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T)
\]

and the purpose of this section is to list sufficient criteria for this equation to possess a unique solution for any \( f \in L^2(0, T; H^{-1}(\Omega)) \) and \( u^0 \in L^2(\Omega) \).

Rewriting (26) in the weak form, we have that

\[
\int_{\Omega_T} -u^\varepsilon(x, t) v(x) \partial_t c(t) + a\left(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \nabla u^\varepsilon\right) \cdot \nabla v(x) c(t) \, dx dt = (27)
\]

\[
\int_0^T \langle f(t), v \rangle_{H^{-1}(\Omega), H^{1,0}_0(\Omega)} c(t) \, dt
\]
for all \( v \in H^1_0(\Omega) \) and \( c \in D(0,T) \), where the solution is searched for in \( H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega)) \) and \( u^\varepsilon(x,0) = u^0(x) \). Thus, the correspondence to \( A(t) \) in (22) is

\[
A^\varepsilon(t) : H^1_0(\Omega) \rightarrow H^{-1}(\Omega)
\]
defined by

\[
\langle A^\varepsilon(t)(\cdot), \cdot \rangle_{H^{-1}(\Omega),H^1_0(\Omega)} = \int_\Omega a \left( x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \nabla(\cdot) \right) \cdot \nabla(\cdot) \, dx,
\]

which for any \( u \in H^1_0(\Omega) \) creates operators \( A^\varepsilon(t)u \) in \( H^{-1}(\Omega) \).

**Remark 7** Equation (27) is equivalent to a corresponding equality between operators in \( L^2(0,T;H^{-1}(\Omega)) \) acting on test functions \( v \in L^2(0,T;H^1_0(\Omega)) \). The corresponding holds true for general operator equations of the type in Section 2.2.1 acting on some suitable space \( V \) of the kind introduced there instead of \( H^1_0(\Omega) \); see Theorem 30.A. (c) in [ZeiIIB].

We will see that the operators \( A^\varepsilon \) satisfy the conditions (I)-(V) for every fixed \( \varepsilon > 0 \), if the function \( a \) fulfills the structure conditions given below. This means that there is a unique \( u^\varepsilon \in H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega)) \) for every \( \varepsilon > 0 \) such that \( u^\varepsilon(x,0) = u^0(x) \) and (27) holds for all \( v \in H^1_0(\Omega), c \in D(0,T) \), i.e. (26) has a unique weak solution.

We assume that

\[
a : \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^{2N} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N
\]
agrees with the following conditions, where \( 0 < k \leq 1 \) and \( \alpha \) and \( \beta \) are positive constants:

(i) \( a(x,t,y_1,y_2,s,0) = 0 \) for all \( (x,t,y_1,y_2,s) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^{2N} \times \mathbb{R} \).

(ii) \( a(\cdot,\cdot,\cdot,\cdot,\xi) \) is \( \mathcal{V}_{2,1} \)-periodic in \( (y_1,y_2,s) \) and continuous for all \( \xi \in \mathbb{R}^N \).

(iii) \( a(x,t,y_1,y_2,s,\cdot) \) is continuous for all \( (x,t,y_1,y_2,s) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^{2N} \times \mathbb{R} \).

(iv) \( |a(x,t,y_1,y_2,s,\xi) - a(x,t,y_1,y_2,s,\xi')| \leq \beta (1 + |\xi| + |\xi'|)^{1-k} |\xi - \xi'|^k \)

for all \( (x,t,y_1,y_2,s) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^{2N} \times \mathbb{R} \) and all \( \xi, \xi' \in \mathbb{R}^N \).

(v) \( (a(x,t,y_1,y_2,s,\xi) - a(x,t,y_1,y_2,s,\xi')) \cdot (\xi - \xi') \geq \alpha |\xi - \xi'|^2 \)

for all \( (x,t,y_1,y_2,s) \in \mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}^{2N} \times \mathbb{R} \) and all \( \xi, \xi' \in \mathbb{R}^N \).
Theorem 8 If $a$ satisfies (i)-(v), then the problem (26) possesses a unique solution $u^\varepsilon \in H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega))$ for every fixed $\varepsilon > 0$.

Proof. We will show that $A^\varepsilon$ satisfies the conditions (I)-(V) in Section 2.2.1 for every $\varepsilon > 0$ and hence the problem (26) will have a unique solution by Theorem 5.

To show that condition (I) is satisfied, we use the knowledge that $a$ fulfills condition (v), which means that

$$\int_{\Omega} \left( a \left( x,t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u \right) - a \left( x,t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla v \right) \right) \cdot \nabla (u(x) - v(x)) \, dx \geq 2 \alpha \int_{\Omega} |\nabla u(x) - \nabla v(x)|^2 \, dx = \alpha \|u - v\|^2_{H^1_0(\Omega)} \geq 0$$

and hence

$$\langle A^\varepsilon(t)u - A^\varepsilon(t)v, u - v \rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \geq 0$$

for all $u, v \in H^1_0(\Omega)$.

From (iv) it follows that

$$\int_{\Omega} \left( a \left( x,t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u + \delta \nabla v \right) - a \left( x,t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u + \delta' \nabla v \right) \right) \cdot \nabla w(x) \, dx \leq$$

$$\int_{\Omega} \beta \left( 1 + |\nabla u(x) + \delta \nabla v(x)| + |\nabla u(x) + \delta' \nabla v(x)| \right)^{1-k} |(\delta - \delta') \nabla v(x)|^k |\nabla w(x)| \, dx$$

which tends to zero as $\delta - \delta'$ does, and thus condition (II) is satisfied.

To prove that condition (III) is fulfilled, we make use of (28). Choosing $v = 0$ and using (i), we get

$$\langle A^\varepsilon(t)u - A^\varepsilon(t)0, u - 0 \rangle_{H^{-1}(\Omega),H^1_0(\Omega)} = \int_{\Omega} \left( a \left( x,t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u \right) - a \left( x,t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, 0 \right) \right) \cdot \nabla u(x) \, dx =$$

$$\int_{\Omega} a \left( x,t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u \right) \cdot \nabla u(x) \, dx \geq \alpha \|u\|^2_{H^1_0(\Omega)}.$$
that is
\[ \langle A^\varepsilon(t)u, u \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \geq \alpha \| u \|_{H^1_0(\Omega)}^2 \]
and hence the operator is coercive.

\( A^\varepsilon \) also satisfies condition (IV). First, we observe that
\[
\| A^\varepsilon(t)u \|_{H^{-1}(\Omega)} = \sup_{\| v \|_{H^1_0(\Omega)} \leq 1} \left| \langle A^\varepsilon(t)u, v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \right| = \\
\sup_{\| v \|_{H^1_0(\Omega)} \leq 1} \left| \int_\Omega a \left( x, t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u \right) \cdot \nabla v(x) \, dx \right| \leq \\
\sup_{\| v \|_{H^1_0(\Omega)} \leq 1} \int_\Omega \left| a \left( x, t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u \right) \right| |\nabla v(x)| \, dx.
\]

From (iv) it follows, by choosing \( \xi' = 0 \) and using (i), that
\[
|a(x, t, y_1, y_2, s, \xi)| \leq \beta (1 + |\xi|^k) (1 + |\xi|)^k < \beta (1 + |\xi|)^{1-k} (1 + |\xi|)^k
\]
and hence
\[
|a(x, t, y_1, y_2, s, \xi)| \leq \beta (1 + |\xi|).
\]

Thus, by (29) and the Hölder inequality we get
\[
\sup_{\| v \|_{H^1_0(\Omega)} \leq 1} \int_\Omega \left| a \left( x, t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u \right) \right| |\nabla v(x)| \, dx \leq \\
\sup_{\| v \|_{H^1_0(\Omega)} \leq 1} \int_\Omega \beta (1 + |\nabla u(x)|) |\nabla v(x)| \, dx \leq \\
\sup_{\| v \|_{H^1_0(\Omega)} \leq 1} \beta \| 1 + |\nabla u| \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)N} = \beta \| 1 + |\nabla u| \|_{L^2(\Omega)}.
\]

Hence, by the triangle inequality,
\[
\| A^\varepsilon(t)u \|_{H^{-1}(\Omega)} < \beta \| 1 + |\nabla u| \|_{L^2(\Omega)} \leq \\
\beta \left( \| 1 \|_{L^2(\Omega)} + \| \nabla u \|_{L^2(\Omega)N} \right) = \beta_1 + \beta \| u \|_{H^1_0(\Omega)}
\]
and since \( \beta \) and \( \beta_1 \) are non-negative constants, condition (IV) is met.
From the continuity assumptions on $a$ and the growth condition (29), it follows that $a(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^{2}}, \nabla u)$ belongs to $L^{2}(\Omega)$. Thus, for $v \in H^{1}_{0}()$

$$g_{3}(t) = \int_{\Omega} a(x, t, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^{2}}, \nabla u) \cdot \nabla v(x) \, dx$$

is in $L^{1}(0, T)$ and hence obviously measurable, which means that condition (V) is fulfilled. □

**Remark 9** We have established the existence of a unique solution to (11) for the possibly nonlinear monotone cases (12) and (13). Case (14) can easily be included in the theorem and proof above, just by adding a third temporal scale. However, for this case linear theory is enough. Consider the linear parabolic system

$$\begin{align*}
\partial_{t} u(x, t) - \nabla \cdot (a(x, t) \nabla u(x, t)) &= f(x, t) \quad \text{in } \Omega, \\
u(x, 0) &= u^{0}(x) \quad \text{in } \Omega, \\
u(x, t) &= 0 \quad \text{on } \partial \Omega \times (0, T),
\end{align*}$$

where $f \in L^{2}(0, T; H^{-1}(\Omega))$ and $u^{0} \in L^{2}(\Omega)$. Here, the conditions are made directly on the coefficient and are the following. We require that $a \in L^{\infty}(\Omega) \times R^{2}$ and satisfies the coercivity condition

$$a(x, t) \xi \cdot \xi \geq \alpha |\xi|^2$$

for some $\alpha > 0$, a.e. $(x, t) \in \Omega$ and all $\xi \in R^{N}$. Under these assumptions, (30) has a unique solution $u \in H^{1}(0, T; H^{1}_{0}(\Omega), H^{-1}(\Omega))$; see Proposition 23.30 in [ZeiIIA].

Studying problem (11) with $a^{\varepsilon}$ given by (14), we have a problem of the form (30) with the coefficient $a\left(\frac{y}{\varepsilon}, \frac{t}{\varepsilon^{2}}, \frac{a^{\varepsilon}}{\varepsilon^{3}}\right)$. We assume that the function

$$a : R^{N} \times R^{2} \rightarrow R^{N \times N}$$

belongs to $L^{\infty}(\mathcal{Y}_{1, 2})_{N \times N}$ and fulfills

$$a(y, s_{1}, s_{2}) \xi \cdot \xi \geq \alpha |\xi|^2$$

for some $\alpha > 0$, all $(y, s_{1}, s_{2}) \in R^{N} \times R^{2}$ and all $\xi \in R^{N}$. From this it is easily seen that the conditions above are satisfied for our coefficient $a\left(\frac{y}{\varepsilon}, \frac{t}{\varepsilon^{2}}, \frac{a^{\varepsilon}}{\varepsilon^{3}}\right)$ with oscillations depending on $\varepsilon$, for every $\varepsilon > 0$. Hence, we have existence and uniqueness of a solution $u^{\varepsilon} \in H^{1}(0, T; H^{1}_{0}(\Omega), H^{-1}(\Omega))$ to (11) also for case (14).
Remark 10  Monotone elliptic and parabolic problems of the kind considered in this chapter have been carefully studied by Zeidler in [ZeiIIA] and [ZeiIIB] for both linear and nonlinear cases.
3 \textit{G}-convergence

The mathematical discipline of homogenization dates back to the late 1960s when Spagnolo introduced a type of convergence of operators called \textit{G}-convergence. As noted in Chapter 1, homogenization is a special case of \textit{G}-convergence. Before we discuss this concept of convergence in more mathematical terms, let us consider an illustration, which highlights the sense in which \textit{G}-convergence is more general than periodic homogenization.

The sequence of equations studied in Section 1.1 could be regarded as describing a phenomenon in a series of materials with a successively finer periodic microstructure. In a more general context we can imagine that we have a sequence of materials that are not periodically built up, but where a similar kind of stabilization takes place. This could for example look like the case illustrated in Figure 8.

![Figure 8. Stabilization towards a limit structure.](image)

Thus, except for omitting the periodicity assumption, we can allow sequences such that the limit structure is not homogeneous.

Let us now make a little more precise what we mean by convergence of operators. To this end, we consider a sequence of equations of the form

\[ A^h u^h = f \]

with unique solutions \( u^h \) belonging to some function space \( X \). For a sequence of operators \( A^h \), we obtain a corresponding sequence of solutions \( u^h \) to these equations. If we look at this from a slightly different angle, this means that we have a sequence of operators \( A^h \) that determine a sequence of functions \( u^h \) such that the same image \( f \) is obtained. If the sequence \( \{u^h\} \) stabilizes to a limit \( u \) in \( X \), i.e. if

\[ u^h \rightarrow u \]

in some reasonable sense, one might wonder if there is an operator \( B \) that produces the same image \( f \) of this limit function \( u \), i.e. if for some \( B \)

\[ Bu = f. \] (31)
If this is the case with the same $B$ for any $f$ that may be in question, $B$ could be seen as the limit of $\{A^h\}$ regarding the capability of transforming functions in $X$, i.e. in the sense of operators.

When searching for the limit equation solved by the limit function $u$, we have to deal with two issues. The first one is to find assumptions under which there exists a limit $B$, with properties such that (31) has a unique solution. The second difficulty is the one of determining $B$, and has been studied mainly for periodic problems. In this chapter, we deal primarily with the first issue.

Although it has been defined for abstract operators, $G$-convergence is mostly considered for partial differential operators, for which it was originally introduced. This will also be the interest in the remainder of this chapter. In Section 3.1, we define the notion of $G$-convergence for elliptic operators and see some results on $G$-convergence compactness, which assure us of the existence of a well-posed limit problem for certain sequences of elliptic PDEs. In Section 3.2, we do the corresponding for parabolic operators.

**Remark 11** For elliptic and parabolic PDEs, we may consider $G$-convergence as convergence of sequences of operators of the types

$$A^h(\cdot) = -\nabla \cdot a^h(x, \nabla(\cdot))$$

and

$$A^h(\cdot) = \partial_t(\cdot) - \nabla \cdot a^h(x, t, \nabla(\cdot)),$$

respectively, in the sense described in the reasoning above. For such equations this is, however, often expressed as that the sequence of functions $a^h$ $G$-converges, or the sequence of coefficients in $a^h$ if $a^h$ is linear. The operator notion is perhaps the most consistent one since it is not restricted to PDEs of this type. However, since the differential operators above are determined by $a^h$, there should be no confusion about which operators are in question if we speak in terms of $a^h$, as long as we know whether we are in the elliptic or the parabolic setting. Also, since this is more convenient, we will use this way of expressing ourselves in what follows.

### 3.1 $G$-convergence for elliptic operators

In this section we start by discussing linear elliptic problems regarding the prospect of finding a limit in the sense described above. We recall the definition of $G$-convergence for such problems and some results concerning the
assumptions under which there exists a $G$-limit. We also discuss some properties of this limit. Then we consider $G$-convergence for possibly nonlinear monotone elliptic problems, where similar conditions can also be identified.

3.1.1 Linear elliptic operators

Let us consider a sequence of equations of the type

$$
-\nabla \cdot (a_h(x) \nabla u^h(x)) = f(x) \quad \text{in } \Omega,
$$

$$
u^h(x) = 0 \quad \text{on } \partial \Omega,
$$

where we let $f \in L^2(\Omega)$, and $a^h$ is assumed to fulfill conditions that guarantee the existence of a unique solution and also an appropriate a priori estimate for the sequence of solutions. To be precise, we assume that the coefficients are functions

$$
a : \Omega \rightarrow \mathbb{R}^{N \times N}
$$

satisfying the following conditions on boundedness and coercivity, where $\alpha$ and $\beta$ are constants such that $0 < \alpha \leq \beta < \infty$:

(M1) $a \in L^\infty(\Omega)^{N \times N}$.

(M2) $|a(x)\xi| \leq \beta |\xi|$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

(M3) $a(x)\xi \cdot \xi \geq \alpha |\xi|^2$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

We name the set of such functions $\mathcal{M}(\alpha, \beta, \Omega)$.

From (M3) it follows that the sequence of solutions $u^h$ is bounded in $H^1_0(\Omega)$, that is,

$$
\|u^h\|_{H^1_0(\Omega)} \leq C
$$

for some positive constant $C$; see Theorem 4.16 in [CiDo]. Hence, there is a function $u$ such that, for a subsequence,

$$
u^h(x) \rightharpoonup u(x) \quad \text{in } H^1_0(\Omega).
$$

If there is a function $b$ such that $u$ uniquely solves the problem

$$
-\nabla \cdot (b(x) \nabla u(x)) = f(x) \quad \text{in } \Omega,
$$

$$
u(x) = 0 \quad \text{on } \partial \Omega,
$$

we could regard $b$ as the limit of this subsequence of $\{a^h\}$ in the capacity of representing an operator, following the more abstract reasoning in the introduction to this chapter.
We recall that we are actually studying the weak form of (32), i.e.
\[
\int_{\Omega} a^h(x) \nabla u^h(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx
\]
(33)
for all \( v \in H^1_0(\Omega) \). Thus, we are looking for a limit problem of the form
\[
\int_{\Omega} b(x) \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx.
\]
(34)

**Remark 12** There would be nothing to hinder us from passing to the limit in (33) if
\[
a^h(x) \to a(x) \quad \text{in } L^\infty(\Omega)^{N \times N}.
\]
We can then easily deduce that
\[
a^h(x) \nabla u^h(x) \rightharpoonup a(x) \nabla u(x) \quad \text{in } L^2(\Omega)^N.
\]

Hence, \( u \) solves the problem
\[
\int_{\Omega} a(x) \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx
\]
for all \( v \in H^1_0(\Omega) \). Since \( a \) belongs to \( \mathcal{M}(\alpha, \beta, \Omega) \) (see Remark 6.1 in [Def] and Remark 5.1 in [CiDo]), \( u \) is the unique solution. Thus, in this case we simply have that \( b = a \).

From the properties of \( a^h \), we can only establish weak* convergence in \( L^\infty(\Omega)^{N \times N} \). Thus, in general we face the same problem as in Section 1.2, that we have a product of two only weakly convergent sequences, and hence we cannot proceed in as straightforward a manner as in Remark 12.

What we can say is that the sequence of products has a weak \( L^2(\Omega)^N \)-limit. From the a priori estimate of \( \{u^h\} \) and the condition \((M2)\), it follows that \( \{a^h(x) \nabla u^h(x)\} \) is bounded in \( L^2(\Omega)^N \) (see Section 5.1 in [CiDo]) and thus there is some \( \sigma \in L^2(\Omega)^N \) such that, up to a subsequence,
\[
a^h(x) \nabla u^h(x) \rightharpoonup \sigma(x) \quad \text{in } L^2(\Omega)^N.
\]
Hence, for such a subsequence we can pass to the limit in (33) obtaining
\[
\int_{\Omega} \sigma(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx
\]
for all \( v \in H^1_0(\Omega) \).
What remains to be investigated is whether we can express $\sigma$ in terms of $u$ in an appropriate way, i.e., if for some $b$

$$\sigma (x) = b (x) \nabla u (x),$$

where $b$ is of the same character as $a^h$, so that there is in fact a well-posed limit problem of the desired form (34).

The first important results on this topic were given by Spagnolo in [Spa1] and [Spa2], where $G$-convergence was introduced for linear elliptic and parabolic operators with symmetric coefficients. We will reproduce a result for the linear elliptic case which shows that, for such operators, the assumption that $a^h$ belongs to ${\mathcal M}(\alpha, \beta, \Omega)$ is actually enough to secure the existence of a well-posed limit problem, at least for a subsequence. Thus, following Spagnolo, we first consider problems where the coefficients $a^h$ are symmetric, i.e.

$$a^h_{ij}(x) = a^h_{ji}(x)$$

for all $x \in \Omega$. Let us also state the exact definition of $G$-convergence for such matrices.

**Definition 13** A sequence $\{a^h\}$ of symmetric matrices belonging to ${\mathcal M}(\alpha, \beta, \Omega)$ is said to $G$-converge to the symmetric matrix $b \in {\mathcal M}(\alpha', \beta', \Omega)$ (called the $G$-limit) if, for every $f \in H^{-1}(\Omega)$, the sequence $\{u^h\}$ of solutions to the equations

$$-\nabla \cdot (a^h(x) \nabla u^h(x)) = f(x) \quad \text{in } \Omega,$$

$$u^h(x) = 0 \quad \text{on } \partial \Omega,$$

satisfies

$$u^h(x) \rightharpoonup u(x) \quad \text{in } H^1_0(\Omega),$$

where $u$ is the unique solution to

$$-\nabla \cdot (b(x) \nabla u(x)) = f(x) \quad \text{in } \Omega,$$

$$u(x) = 0 \quad \text{on } \partial \Omega.$$

The following result on the compactness of the set of symmetric matrices in $\mathcal{M}(\alpha, \beta, \Omega)$ with respect to $G$-convergence is proven in [Spa2].

**Theorem 14** Any sequence $\{a^h\}$ of symmetric matrices in $\mathcal{M}(\alpha, \beta, \Omega)$ possesses a subsequence $G$-converging to some limit $b$ in the same set.
Under the name of $H$-convergence, a generalization of $G$-convergence to the non-symmetric case has been made by Murat and Tartar. The definition of $H$-convergence requires—besides (35), i.e. the weak convergence of $\{u^h\}$ in $H^1_0(\Omega)$—that

$$a^h(x) \nabla u^h(x) \rightharpoonup b(x) \nabla u(x) \quad \text{in } L^2(\Omega)^N. \quad (36)$$

The reason is that if we omit the symmetry assumption in Definition 13, it may happen that the limit $b$ is not unique. An illustrative example of this can be found in [Def]. For a $G$-convergent sequence of symmetric matrices the property (36) always holds true, and for such matrices $G$- and $H$-convergence are equivalent.

**Remark 15** Since $H$-convergence is a natural generalization of $G$-convergence rather than a different kind of convergence, in the remainder of this thesis we will keep to the term $G$-convergence for both symmetric and non-symmetric cases.

The set $\mathcal{M}(\alpha, \beta, \Omega)$, including non-symmetric matrices, is compact with respect to $G$-convergence in the sense given in the following theorem, proven in [MuTa].

**Theorem 16** Any sequence $\{a^h\}$ in $\mathcal{M}(\alpha, \beta, \Omega)$ has a subsequence that $G$-converges to some limit $b$ in $\mathcal{M}(\alpha, \frac{\beta^2}{\alpha}, \Omega)$.

Note that in this case, though preserving the properties of boundedness and coercivity, the limit belongs to a larger set, unless $\alpha = \beta$.

For problems with sequences of coefficients belonging to $\mathcal{M}(\alpha, \beta, \Omega)$, the compactness results mentioned guarantee the existence of a well-posed limit problem, at least for a subsequence. To draw the conclusion that the entire sequence $G$-converges, and not only a subsequence, we need the following theorem; see e.g. [Spa2].

**Theorem 17** A sequence in $\mathcal{M}(\alpha, \beta, \Omega)$ $G$-converges if and only if all $G$-convergent subsequences have the same limit.

The results above can be applied, for example, to periodic problems like the model problem (1). This problem is treated in Sections 4.1, 4.2 and 5.1.2, where we also find an explicit expression for the $G$-limit. In Section 5.2.2, we give a similar result for a periodic problem with two microscopic scales. Another problem with coefficients in $\mathcal{M}(\alpha, \beta, \Omega)$, and without periodicity assumptions, is investigated in Section 6.1.
In Remark 12, we saw a situation where we could immediately determine the $G$-limit. A similar result can also be obtained under some more modest assumptions, as stated in the proposition below.

**Proposition 18** Let \( \{a^h\} \) be a sequence in \( \mathcal{M}(\alpha, \beta, \Omega) \). If \( \{a^h\} \) converges to a either in \( L^1(\Omega)^{N \times N} \) or almost everywhere in \( \Omega \), then \( \{a^h\} \) $G$-converges to \( a \).

**Proof.** See the proof of Lemma 1.2.22 in [All3].

We end this section by listing some important results on properties possessed by the $G$-limit. These qualities are required from all types of $G$-convergence in order to obtain a meaningful concept of convergence.

- The limit of a $G$-convergent sequence is unique.
- By definition, the $G$-limit does not depend on the source term \( f \).
- The $G$-limit does not depend on the boundary conditions.
- Assume that \( \{a^h\} \) and \( \{\tilde{a}^h\} \) are two sequences in \( \mathcal{M}(\alpha, \beta, \Omega) \), which $G$-converge to \( b \) and \( \tilde{b} \), respectively. If, for some open subset \( \omega \) in \( \Omega \),
  \[
  a^h(x) = \tilde{a}^h(x) \quad \text{in } \omega, 
  \]
  then
  \[
  b(x) = \tilde{b}(x) \quad \text{in } \omega. 
  \]
- If a sequence \( \{a^h\} \) $G$-converges to a limit \( b \) in \( \Omega \), then \( \{a^h\} \) $G$-converges to the same limit in any subset of \( \Omega \) (referring now to the restrictions to the subset).

**Remark 19** The properties listed above are of interest in the case of computing effective properties. The limit coefficient is unique and the determination can be done once and for all. This is, of course, a desirable quality. We do not want to derive effective properties over and over if we would like to study a problem for different source terms. Nor do we want the derived coefficient to depend on the sample size or on the boundary conditions, following the same reasoning.
3.1.2 Monotone elliptic operators

$G$-convergence can be formulated analogously for possibly nonlinear monotone elliptic problems under certain assumptions on the governing function $a$, such as strong monotonicity and a certain continuity condition. We define $\mathcal{N}(\alpha, \beta, k, \Omega)$ to be the set of functions

$$a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$$

such that, for constants $\alpha$ and $\beta$ satisfying $0 < \alpha \leq \beta < \infty$ and $0 < k \leq 1$, it holds that:

(N1) $a(x,0) = 0$ for a.e. $x \in \Omega$.

(N2) $a(\cdot, \xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^N$.

(N3) $|a(x, \xi) - a(x, \xi')| \leq \beta (1 + |\xi| + |\xi'|)^{1-k} |\xi - \xi'|^k$
for a.e. $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$.

(N4) $(a(x, \xi) - a(x, \xi')) \cdot (\xi - \xi') \geq \alpha |\xi - \xi'|^2$
for a.e. $x \in \Omega$ and for all $\xi, \xi' \in \mathbb{R}^N$.

We are now prepared for the following definition.

**Definition 20** A sequence $\{a^h\}$ in $\mathcal{N}(\alpha, \beta, k, \Omega)$ is said to $G$-converge to $b \in \mathcal{N}(\alpha', \beta', k', \Omega)$ if, for every $f \in H^{-1}(\Omega)$, the sequence $\{u^h\}$ of solutions to the equations

$$-\nabla \cdot a^h(x, \nabla u^h) = f(x) \quad \text{in } \Omega,$$

$$u^h(x) = 0 \quad \text{on } \partial \Omega,$$

satisfies

$$u^h(x) \rightharpoonup u(x) \quad \text{in } H^1_0(\Omega),$$

$$a^h(x, \nabla u^h) \rightharpoonup b(x, \nabla u) \quad \text{in } L^2(\Omega)^N,$$

where $u$ is the unique solution to

$$-\nabla \cdot b(x, \nabla u) = f(x) \quad \text{in } \Omega,$$

$$u(x) = 0 \quad \text{on } \partial \Omega.$$
For $k = 1$, the condition $(N3)$ becomes more transparent and takes the form
\[ |a(x, \xi) - a(x, \xi')| \leq \beta |\xi - \xi'| \]
and we denote the corresponding set of functions by $\mathcal{N}(\alpha, \beta, \Omega)$. This is the class that was studied by Tartar in the original result for possibly nonlinear monotone problems. The theorem below is proven in [Tar1].

**Theorem 21** Every sequence $\{a^h\}$ belonging to $\mathcal{N}(\alpha, \beta, \Omega)$ $G$-converges, up to a subsequence, to some $b$ in $\mathcal{N}(\alpha, \beta^2/\alpha, \Omega)$.

A similar compactness result for the set $\mathcal{N}(\alpha, \beta, k, \Omega)$ can be concluded from Lemma 2.3.2 in [Pan].

**Theorem 22** Any sequence $\{a^h\}$ in $\mathcal{N}(\alpha, \beta, k, \Omega)$ has a subsequence that $G$-converges to some $b \in \mathcal{N}(\bar{\alpha}, \bar{\beta}, \bar{k}, \Omega)$, where $\bar{k} = k/(2 - k)$ and $\bar{\alpha}$ and $\bar{\beta}$ are positive constants depending only on $\alpha$ and $\beta$.

In Section 4.2, we study the monotone correspondence to the model problem with $a^h$ in $\mathcal{N}(\alpha, \beta, \Omega)$, where the $G$-limit is determined by classical homogenization techniques. In Section 4.3, we give an account of a result where the $G$-limit is characterized for another periodic problem with $a^h$ in $\mathcal{N}(\alpha, \beta, k, \Omega)$ and where oscillations occur on two microscopic scale levels.

**Remark 23** $G$-convergence of linear elliptic operators is investigated in e.g. [Spa2] and [Spa3] by Spagnolo, in [GiSp] by De Giorgi and Spagnolo, and by Jikov et al. in [JKO]. The nonlinear case was first studied by Tartar in [Tar1]. A reproduction of the proof of the compactness result can also be found in [Del]. $G$-convergence of nonlinear monotone, possibly multivalued operators is treated by Chiadò Piat et al. in [CDD] and [ChDe]. For some fundamental results on $H$-convergence and its theoretical background, we refer to [Mur1] and [Mur2] by Murat, [Tar1] and [Tar2] by Tartar, and [MuTa] by Murat and Tartar. See also the book [Eva] by Evans.

### 3.2 $G$-convergence for parabolic operators

The notion of $G$-convergence as defined in Section 3.1 can also be extended to parabolic problems. As we did for elliptic operators, we first consider linear problems and then we proceed with possibly nonlinear monotone problems. Finally, we consider a result which, under certain assumptions, allows us to obtain the $G$-limit in the sense of parabolic operators if we know the $G$-limit in the elliptic sense.
3.2.1 Linear parabolic operators

As in the linear elliptic case, we first define the set of coefficients we will consider. Let $\mathcal{M}(\alpha, \beta, \Omega_T)$ be the set of functions

$$a : \Omega_T \to \mathbb{R}^{N \times N}$$

satisfying the following conditions, where $\alpha$ and $\beta$ are constants such that $0 < \alpha \leq \beta < \infty$:

(M1) $a \in L^\infty(\Omega_T)^{N \times N}$.

(M2) $|a(x,t)\xi| \leq \beta |\xi|$ for a.e. $(x,t) \in \Omega_T$ and for all $\xi \in \mathbb{R}^N$.

(M3) $a(x,t)\xi \cdot \xi \geq \alpha |\xi|^2$ for a.e. $(x,t) \in \Omega_T$ and for all $\xi \in \mathbb{R}^N$.

$G$-convergence for linear parabolic operators is defined as follows.

**Definition 24** A sequence $\{a^h\}$ in $\mathcal{M}(\alpha, \beta, \Omega_T)$ is said to $G$-converge to $b \in \mathcal{M}(\alpha', \beta', \Omega_T)$ if, for every $f \in L^2(0,T;H^{-1}(\Omega))$ and $u^0 \in L^2(\Omega)$, the sequence $\{u^h\}$ of solutions to the equations

$$\partial_t u^h(x,t) - \nabla \cdot (a^h(x,t) \nabla u^h(x,t)) = f(x,t) \text{ in } \Omega_T,$$

$$u^h(x,0) = u^0(x) \text{ in } \Omega,$$

$$u^h(x,t) = 0 \text{ on } \partial \Omega \times (0,T),$$

satisfies

$$u^h(x,t) \rightharpoonup u(x,t) \text{ in } L^2(0,T;H^1_0(\Omega)),$$

$$a^h(x,t) \nabla u^h(x,t) \rightharpoonup b(x,t) \nabla u(x,t) \text{ in } L^2(\Omega_T)^N,$$

where $u$ is the unique solution to

$$\partial_t u(x,t) - \nabla \cdot (b(x,t) \nabla u(x,t)) = f(x,t) \text{ in } \Omega_T,$$

$$u(x,0) = u^0(x) \text{ in } \Omega,$$

$$u(x,t) = 0 \text{ on } \partial \Omega \times (0,T).$$

The definition is motivated by the following compactness result for the set $\mathcal{M}(\alpha, \beta, \Omega_T)$; see [Spa4] and [Sva1].

**Theorem 25** A sequence $\{a^h\}$ belonging to $\mathcal{M}(\alpha, \beta, \Omega_T)$ possesses a subsequence that $G$-converges to some $b$ in $\mathcal{M}(\alpha, \beta^2/\alpha, \Omega_T)$.

This will be applied to a periodic problem with two scales in space and three scales in time in Section 5.3.3, and to a problem without periodicity assumptions in Section 6.2.
3.2.2 Monotone parabolic operators

For our purposes, we also need to define $G$-convergence of parabolic operators for the monotone possibly nonlinear case, which we formulate following Svanstedt; see [Sva1]. We define $\mathcal{N}(\alpha, \beta, k, \Omega_T)$ as the set of functions

$$a : \Omega_T \times \mathbb{R}^N \rightarrow \mathbb{R}^N$$

satisfying the following conditions, where $0 < k \leq 1$ and $\alpha$ and $\beta$ are positive constants:

$(N1)$ $a(x, t, 0) = 0$ for a.e. $(x, t) \in \Omega_T$.

$(N2)$ $a(\cdot, \cdot, \xi)$ is Lebesgue measurable for every $\xi \in \mathbb{R}^N$.

$(N3)$ $|a(x, t, \xi) - a(x, t, \xi')| \leq \beta (1 + |\xi| + |\xi'|)^{1-k} |\xi - \xi'|^k$

for a.e. $(x, t) \in \Omega_T$ and for all $\xi, \xi' \in \mathbb{R}^N$.

$(N4)$ $(a(x, t, \xi) - a(x, t, \xi')) \cdot (\xi - \xi') \geq \alpha |\xi - \xi'|^2$

for a.e. $(x, t) \in \Omega_T$ and for all $\xi, \xi' \in \mathbb{R}^N$.

The definition of $G$-convergence for the monotone parabolic case is given below.

**Definition 26** A sequence $\{a^h\}$ in $\mathcal{N}(\alpha, \beta, k, \Omega_T)$ is said to $G$-converge to $b \in \mathcal{N}(\alpha', \beta', k', \Omega_T)$ if, for every $f \in L^2(0, T; H^{-1}(\Omega))$ and $u^0 \in L^2(\Omega)$, the sequence $\{u^h\}$ of solutions to the equations

$$\partial_t u^h(x, t) - \nabla \cdot a^h(x, t, \nabla u^h) = f(x, t) \quad \text{in } \Omega_T,$$

$$u^h(x, 0) = u^0(x) \quad \text{in } \Omega,$$

$$u^h(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T),$$

satisfies

$$u^h(x, t) \rightharpoonup u(x, t) \quad \text{in } L^2(0, T; H^1_0(\Omega)),$$

$$a^h(x, t, \nabla u^h) \rightharpoonup b(x, t, \nabla u) \quad \text{in } L^2(\Omega_T)^N,$$

where $u$ is the unique solution to

$$\partial_t u(x, t) - \nabla \cdot b(x, t, \nabla u) = f(x, t) \quad \text{in } \Omega_T,$$

$$u(x, 0) = u^0(x) \quad \text{in } \Omega,$$

$$u(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T).$$
As proven in [Sva1], the set \( N(\alpha, \beta, k, \Omega_T) \) is sequentially compact in the sense given in the following theorem.

**Theorem 27** A sequence \( \{a^h\} \) in \( N(\alpha, \beta, k, \Omega_T) \) possesses a subsequence that \( G \)-converges to a limit \( b \in N(\alpha, \beta, k, \Omega_T) \), where \( \bar{\beta} = k/(2 - k) \) and \( \bar{\beta} \) is a positive constant depending on \( \alpha \), \( \beta \) and \( k \) only.

**Proof.** See the proof of Theorem 5.2 in [Sva1].

This will be used for example in Section 4.3, where we prove \( G \)-convergence for a monotone parabolic problem with multiple scales in space. A similar problem with several scales in both time and space is investigated in Section 5.3.2.

The \( G \)-limits for parabolic operators will have properties essentially corresponding to those of the limit in the linear elliptic case, listed in Section 3.1. For the parabolic case, we can also add the independence of the time interval; see [Sva1] and [Sva2].

So far, we have considered some fundamental results on existence and properties of the \( G \)-limit for sequences of elliptic and parabolic operators. There exist also results on comparisons between the \( G \)-limits in these two cases. These make it possible to obtain the \( G \)-limit for a parabolic case once the corresponding elliptic result has been established, provided that we make a certain continuity assumption for the \( t \)-dependence. We have the following result for monotone problems.

**Theorem 28** Let \( B : \mathbb{R}_+ \to \mathbb{R}_+ \) be an increasing continuous function such that \( B(t) \to 0 \) as \( t \to 0_+ \). Assume that

\[
|a^h(x, t, \xi) - a^h(x, t', \xi)| \leq B(t - t')(1 + |\xi|) \tag{37}
\]

for all \( \xi \in \mathbb{R}^N \), a.e. \( x \in \Omega \) and all \( t \) and \( t' \) such that \( 0 < t' < t < T \). Assume also that \( \{a^h\} \) \( G \)-converges to \( b \) in the sense of parabolic operators, and that \( \{a^h(t)\} \) \( G \)-converges to \( b'(t) \) in the sense of elliptic operators for every fixed \( t \in (0, T) \). Then \( b = b' \).

**Proof.** See Theorem 6.6 in [Sva1].

We will return to this in Section 4.3, where we use this result to prove homogenization of the parabolic problem with several spatial scales mentioned above. In this connection, we also need the following theorem on \( G \)-convergence for parameter-dependent elliptic problems, that is, elliptic problems where \( t \) appears in \( a^h \) in the role of a parameter.
**Theorem 29** Let $B$ be as in Theorem 28. Assume that $\{a^h(t)\}$ belongs to $\mathcal{N}(\alpha, \beta, k, \Omega)$ for every fixed $t \in (0, T)$ and that

$$|a^h(x, t, \xi) - a^h(x, t', \xi)| \leq B(t - t')(1 + |\xi|)$$

for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$ and all $t$ and $t'$ such that $0 < t' < t < T$. Then, there exists a subsequence such that $\{a^h(t)\}$ $G$-converges to a function $b(t)$, with the same subsequence for all $t \in (0, T)$.

**Proof.** See Theorem 4.2 in [Sva1].

**Remark 30** Results of comparisons for the linear case, similar to the one in Theorem 28, can be found in [CoSp]. Similar results are also contained in [Par]. A linear correspondence to Theorem 29 is given in [CoSp].

**Remark 31** Parabolic $G$-convergence for linear problems is studied by Spagnolo in [Spa1], [Spa2] and [Spa4], and by Colombini and Spagnolo in [CoSp]. See also [ZKO] by Zhikov et al. Results on $G$-convergence for the possibly nonlinear monotone parabolic case are proven by Svanstedt in [Sva1] and [Sva2], and in [Pan] by Pankov.
4 Classical homogenization techniques

For problems with \( \{a^h\} \) in the sets defined in the preceding chapter, we are guaranteed \( G \)-convergence up to a subsequence, i.e. the existence of a limit problem for this subsequence, where the \( G \)-limit \( b \) has the desired properties. There are, however, not many clues as to how to compute this limit. In the remainder of the thesis we will proceed with this issue and develop techniques for the characterization of the limit operator. In this chapter we will see a couple of classical methods to find \( b \) explicitly in periodic problems. In other words, we now turn our interest to periodic homogenization.

Let us again consider the model problem

\[
\begin{align*}
-\nabla \cdot \left( a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon (x) \right) &= f(x) \quad \text{in } \Omega, \\
u^\varepsilon (x) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(38)

where we assume that \( a \in \mathcal{M}(\alpha, \beta, \mathbb{R}^N) \) is \( Y \)-periodic and \( f \in L^2(\Omega) \). Under these assumptions \( a(x/\varepsilon) \) belongs to \( \mathcal{M}(\alpha, \beta, \Omega) \). From Theorem 16 we then know that, at least for a subsequence, we have \( G \)-convergence and hence a limit problem of the form

\[
\begin{align*}
-\nabla \cdot (b(x)\nabla u(x)) &= f(x) \quad \text{in } \Omega, \\
u(x) &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(39)

with \( b \in \mathcal{M}(\alpha, \frac{\beta^2}{\alpha}, \Omega) \). The well-known asymptotic expansion method allows us to proceed one step further, in the sense that we can obtain formally both the local and the homogenized problem and hence the limit matrix \( b \). A sketch of this method is given in Section 4.1. This is, however, still only a guess and is not proven in a mathematically rigorous way. In Section 4.2 we show how the proof is done by a classical method put forward by Tartar, known as the oscillating test function method.

These methods can also be applied to equations with periodic oscillations on several microscopic scales, that is, reiterated homogenization problems. Such problems are considered in Section 4.3. Here we recall a homogenization result for an elliptic problem with two micro-scales. We then study the corresponding parabolic case, i.e. problem (11) with \( a^\varepsilon \) given by (12). We prove the existence of a \( G \)-limit for this problem, and we also characterize the limit operator by using the homogenization result for elliptic operators and a comparison theorem for \( G \)-limits.
4.1 The method of asymptotic expansions

The method of asymptotic expansions is based on the assumption that the solution $u^\varepsilon$ to (38) can be expanded in a power series in $\varepsilon$ of the form

$$u^\varepsilon(x) = u_0 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon u_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon^2 u_2 \left( x, \frac{x}{\varepsilon} \right) + ..., \quad (40)$$

where each $u_k$ is $Y$-periodic in the second argument. The idea is to substitute the asymptotic expansion into the problem (38) and equate the terms that contain the same powers of $\varepsilon$. This will lead us to the local and the homogenized problem. The ansatz (40) is reasonable in the sense that it takes into account the two different length scales in the problem, i.e. the global and the local. However, we cannot be sure that $u^\varepsilon$ does in fact admit an expansion of this kind, which means that the validity of the result has to be proven afterwards.

First, we observe that by using the chain rule the operator

$$A^\varepsilon(\cdot) = -\nabla \cdot \left( a \left( \frac{x}{\varepsilon} \right) \nabla (\cdot) \right)$$

can be written as (see Section 2.2 in [PPSV])

$$A^\varepsilon = \varepsilon^{-2} A_0 + \varepsilon^{-1} A_1 + A_2,$$

where

$$A_0(\cdot) = -\nabla_y \cdot (a (y) \nabla_y (\cdot)), \quad A_1(\cdot) = -\nabla_y \cdot (a (y) \nabla (\cdot)) - \nabla \cdot (a (y) \nabla_y (\cdot)), \quad A_2(\cdot) = -\nabla \cdot (a (y) \nabla (\cdot)).$$

Now, inserting the expansion (40) into (38), we have

$$\varepsilon^{-2} A_0 u_0 + \varepsilon^{-1} (A_0 u_1 + A_1 u_0) + (A_0 u_2 + A_1 u_1 + A_2 u_0) + \varepsilon (A_1 u_2 + A_2 u_1) + ... = f.$$  

Equating equal powers of $\varepsilon$, we get a series of equations, one for each power. The first three become

$$A_0 u_0 = 0, \quad (41)$$
$$A_0 u_1 = -A_1 u_0, \quad (42)$$
$$A_0 u_2 = -A_1 u_1 - A_2 u_0 + f. \quad (43)$$
These problems all have $Y$-periodicity in the second variable as boundary condition. Problems of this kind possess a unique solution, up to an additive constant, if and only if the integral mean value over $Y$ of the right-hand side equals zero; see e.g. Lemma 3.2. in [Def]. Thus equation (41) has a unique solution up to the addition of a function depending only on $x$. Since one solution of (41) is $u_0 = 0$, it follows that

$$u_0 (x, y) = u (x),$$

i.e. that the first term in the expansion actually depends only on $x$. Taking this into account and rearranging equation (42), we get the problem

$$-\nabla_y \cdot (a (y) (\nabla u (x) + \nabla_y u_1 (x, y))) = 0 \quad \text{in } \Omega \times Y.$$  

Using separation of variables, we have

$$u_1 (x, y) = \nabla u (x) \cdot z (y), \quad (44)$$

where $z$ is $Y$-periodic and satisfies

$$-\nabla_y \cdot (a (y) (c_j + \nabla_y z_j (y))) = 0 \quad \text{in } Y. \quad (45)$$

This is the so-called local problem. Finally, using the requirement that the right-hand side should have integral mean value zero over $Y$ in equation (43) and inserting (44), we get

$$-\nabla \cdot (b \nabla u (x)) = f (x) \quad \text{in } \Omega,$$

where $b$ is given by

$$b_{ij} = \int_Y a_{ij} (y) + \sum_{k=1}^N a_{ik} (y) \partial_{y_k} z_j (y) \, dy, \quad (46)$$

which means that we have found the homogenized problem.

This suggests that we are able to compute the limit $b$ by solving PDEs on one representative unit $Y$. Let us exemplify this with a two-dimensional problem, where we choose

$$a(y) = \begin{pmatrix} 2 + 1.95 \sin 2\pi (y_1 + y_2) & 0 \\ 0 & 2 + 1.95 \cos 2\pi (y_1 + y_2) \end{pmatrix},$$
\[ f(x) = x_1^2 \] and \( \Omega = (0, 1)^2 \). Rewriting (45), we obtain the elliptic problems
\[ -\nabla_y \cdot (a(y) \nabla_y z_j(y)) = \nabla_y \cdot (a(y)e_j) \quad \text{in } Y. \tag{47} \]

For our choice of \( a \), we get the right-hand side
\[ \nabla_y \cdot (a(y)e_1) = 2\pi \cdot 1.95 \cos 2\pi(y_1 + y_2) \]
for \( j = 1 \) and
\[ \nabla_y \cdot (a(y)e_2) = -2\pi \cdot 1.95 \sin 2\pi(y_1 + y_2) \]
for \( j = 2 \). Inserting the corresponding solutions \( z_1 \) and \( z_2 \) to (47) in (46), we obtain
\[ b = \begin{pmatrix} 1.4485 & 0 \\ 0 & 1.4485 \end{pmatrix}. \tag{48} \]

If the result above gives the proper homogenized equation, the solutions \( u^\varepsilon \) to (38) for
\[ a \left( \frac{x}{\varepsilon} \right) = \begin{pmatrix} 2 + 1.95 \sin \frac{2\pi(x_1 + x_2)}{\varepsilon} & 0 \\ 0 & 2 + 1.95 \cos \frac{2\pi(x_1 + x_2)}{\varepsilon} \end{pmatrix} \]
should be well approximated by the solution \( u \) to (39) with \( b \) given by (48), if \( \varepsilon \) is small.

![Figure 9. The entries \( a_{11}(\frac{x}{\varepsilon}) \) for \( \varepsilon = 0.1 \) and \( b_{11} \).](image)

A comparison between \( u^\varepsilon \) for \( \varepsilon = 0.1 \) and \( u \) (see Figure 10) confirms this, and for any reasonable right-hand side \( f \) in (38) and (39) the same phenomenon appears.

40
4.2 Tartar’s method of oscillating test functions

The result from the previous section is heuristically obtained. Let us now consider the following theorem, which will be proven in a mathematically rigorous way.

**Theorem 32** Let \( \{u^\varepsilon\} \) be a sequence of solutions in \( H^1_0(\Omega) \) to the sequence of problems

\[
-\nabla \cdot \left( a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon(x) \right) = f(x) \quad \text{in } \Omega,
\]

\[
u^\varepsilon(x) = 0 \quad \text{on } \partial\Omega,
\]

where \( a \in \mathcal{M}(\alpha, \beta, \mathbb{R}^N) \) is \( Y \)-periodic and \( f \in L^2(\Omega) \). Then it holds that

\[
u^\varepsilon(x) \xrightarrow{\ast} u(x) \quad \text{in } H^1_0(\Omega).
\]

The function \( u \in H^1_0(\Omega) \) is the unique solution to

\[
-\nabla \cdot (b \nabla u(x)) = f(x) \quad \text{in } \Omega,
\]

\[
u(x) = 0 \quad \text{on } \partial\Omega,
\]

with

\[
b_{ij} = \int_Y a_{ij}(y) + \sum_{k=1}^N a_{ik}(y) \partial_{y_k} z_j(y) \, dy,
\]

where, for \( j = 1, ..., N \), \( z_j \in H^1_0(Y)/\mathbb{R} \) uniquely solves the local problem

\[
-\nabla_y \cdot (a(y)(e_j + \nabla_y z_j(y))) = 0 \quad \text{in } Y.
\]
In this section we show how the proof is done for the symmetric case using Tartar’s method of oscillating test functions. We will also reproduce a different proof of this theorem in Section 5.1.2 in order to illustrate a more recent homogenization method based on so-called two-scale convergence.

Let us first sum up what we already know. As we noted in the discussion in Section 3.1.1, there are \( u \) and \( \sigma \) such that
\[
\begin{align*}
\nu^\varepsilon (x) & \rightarrow u(x) \quad \text{in } H^1_0(\Omega) \\
\varepsilon \left( \frac{x}{\varepsilon} \right) \nabla \nu^\varepsilon (x) & \rightarrow \sigma(x) \quad \text{in } L^2(\Omega)^N
\end{align*}
\] (53)

and
\[
\begin{align*}
\varepsilon \left( \frac{x}{\varepsilon} \right) \nabla \nu^\varepsilon (x) & \rightarrow \sigma(x) \quad \text{in } L^2(\Omega)^N
\end{align*}
\] (54)

up to a subsequence. Hence, for such a subsequence, there exists a limit problem of the form
\[
\int_{\Omega} \sigma(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx,
\] (55)

holding for all \( v \in H^1_0(\Omega) \). By the G-convergence result, we know that \( \sigma \) can be expressed as
\[
\sigma(x) = b(x) \nabla u(x)
\]

for some \( b \) with the desired properties. This means that what remains to be proven is that \( b \) is in fact the particular one given in (51). If this can be done independently of the subsequence, we have found the homogenized problem.

To prove the result in Theorem 32 rigorously, we want to pass to the limit in the weak form of (49), i.e.
\[
\int_{\Omega} a \left( \frac{x}{\varepsilon} \right) \nabla \nu^\varepsilon (x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx
\] (56)

for any \( v \in H^1_0(\Omega) \). The essential difficulty, which we saw already in the introduction, is that we have the product of two only weakly convergent sequences. The trick is to construct certain test functions that will allow us to study the limit process of (56). In Tartar’s method, the approach for creating these test functions is to make use of an equation that is generated from the local problem (52) and defined on all of \( \Omega \). We then make a crosswise choice of test functions in this problem and (56), in the sense that we let the solutions to the respective problems (multiplied by a smooth function \( \phi \)) act as test functions in the other. Then, subtracting one problem from the other makes the terms that contain products of two only weakly convergent sequences cancel out.
The following two lemmas on extensions of periodic functions, see e.g. the appendix in [Def], will be useful in the remainder of this section.

**Lemma 33** A function \( f \in H_2^1(Y)/\mathbb{R} \) can be extended by periodicity to a function in \( H_1^{\text{loc}}(\mathbb{R}^N) \).

**Lemma 34** Let \( g \in L^2(Y)^N \) and assume that \( \int_Y g \cdot \nabla v \, dy = 0 \) for every \( v \in H_1^1(Y)/\mathbb{R} \). Then \( g \) can be extended by periodicity to a function in \( L_1^{\text{loc}}(\mathbb{R}^N)^N \), still denoted by \( g \), such that \( -\nabla \cdot g = 0 \) in \( D'(\mathbb{R}^N) \).

First, we note that the local problem has a unique solution in \( H_1^1(Y)/\mathbb{R} \). This is realized if we rewrite it in the form

\[-\nabla_y \cdot (a(y)\nabla_y z_j(y)) = \nabla_y \cdot (a(y)e_j) \quad \text{in } Y.
\]

The right-hand side belongs to \((H_2^1(Y)/\mathbb{R})'\) and an application of the Lax-Milgram theorem then gives that there exists a unique solution in \( H_1^1(Y)/\mathbb{R} \); see Theorem 4.26 and Section 7.1 in [CiDo].

Let us now see how to use the local problem to construct appropriate test functions. The weak form of the local problem states that

\[\int_Y a(y)(e_j + \nabla_y z_j(y)) \cdot \nabla_y v(y) \, dy = 0\]

for all \( v \in H_1^1(Y)/\mathbb{R} \). Lemma 33 allows us to make a \( Y \)-periodic extension of \( z \) to a function in \( H_1^{\text{loc}}(\mathbb{R}^N)^N \). Similarly, by Lemma 34 there is a \( Y \)-periodic extension of \( a(y)(e_j + \nabla_y z_j(y)) \) in \( L_1^{\text{loc}}(\mathbb{R}^N)^N \) such that

\[\int_{\mathbb{R}^N} a\left(\frac{x}{\varepsilon}\right)\left(e_j + \nabla_y z_j\left(\frac{x}{\varepsilon}\right)\right) \cdot \nabla v(x) \, dx = 0\]

for all \( v \in D(\mathbb{R}^N) \). Thus, defining

\[w_\varepsilon^j(x) = x_j + \varepsilon z_j\left(\frac{x}{\varepsilon}\right)\]

for \( j = 1, 2, ..., N \) we have

\[\int_{\mathbb{R}^N} a\left(\frac{x}{\varepsilon}\right) \nabla w_\varepsilon^j(x) \cdot \nabla v(x) \, dx = 0\]

for all \( v \in D(\mathbb{R}^N) \), and hence

\[\int_{\Omega} a\left(\frac{x}{\varepsilon}\right) \nabla w_\varepsilon^j(x) \cdot \nabla v(x) \, dx = 0 \quad (57)\]

for any \( v \in H_0^1(\Omega) \).
For \( w_j^\varepsilon \) we have

\[ w_j^\varepsilon (x) \rightarrow x_j \quad \text{in} \quad \mathbb{L}^2(\Omega) \quad (58) \]

and

\[ \nabla w_j^\varepsilon (x) \rightarrow e_j \quad \text{in} \quad \mathbb{L}^2(\Omega)^N. \]

Now, for \( \phi \in D(\Omega) \), we choose test functions \( v = \phi w_j^\varepsilon \) in (56) and \( v = \phi u^\varepsilon \) in (57), and obtain

\[
\int_\Omega a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon(x) \cdot \nabla \phi(x) \, w_j^\varepsilon(x) + a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon(x) \cdot \nabla w_j^\varepsilon(x) \phi(x) \, dx = \int_\Omega f(x) \phi(x) \, w_j^\varepsilon(x) \, dx \tag{59}
\]

and

\[
\int_\Omega a \left( \frac{x}{\varepsilon} \right) \nabla w_j^\varepsilon(x) \cdot \nabla \phi(x) \, u^\varepsilon(x) + a \left( \frac{x}{\varepsilon} \right) \nabla w_j^\varepsilon(x) \cdot \nabla u^\varepsilon(x) \phi(x) \, dx = 0, \tag{60}
\]

respectively. Subtracting (60) from (59) and using the fact that \( a \) is symmetric, we arrive at

\[
\int_\Omega a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon(x) \cdot \nabla \phi(x) \, w_j^\varepsilon(x) - a \left( \frac{x}{\varepsilon} \right) \nabla w_j^\varepsilon(x) \cdot \nabla \phi(x) \, u^\varepsilon(x) \, dx = \int_\Omega f(x) \phi(x) \, w_j^\varepsilon(x) \, dx \tag{61}
\]

Note that through this procedure we get rid of the problematic terms and we are left only with terms that consist of one weakly and one strongly convergent sequence. Indeed, by (54) and (58), we note that the first term in (61) is under control. Furthermore, we have

\[
\left( a \left( \frac{x}{\varepsilon} \right) \nabla w_j^\varepsilon(x) \right)_k = \sum_{i=1}^N a_{ki} \left( \frac{x}{\varepsilon} \right) \partial_{x_i} w_j^\varepsilon(x) = \sum_{i=1}^N a_{ki} \left( \frac{x}{\varepsilon} \right) \left( \delta_{ij} + \partial_{y_i} z_j \left( \frac{x}{\varepsilon} \right) \right)
\]

and thus

\[
\left( a \left( \frac{x}{\varepsilon} \right) \nabla w_j^\varepsilon(x) \right)_k \rightarrow \int_y a_{kj}(y) + \sum_{i=1}^N a_{ki}(y) \partial_{y_i} z_j(y) \, dy = b_{kj} \quad \text{in} \quad \mathbb{L}^2(\Omega)
\]

with \( b \) as in (51). Also, by (53) together with the Rellich embedding theorem, we get

\[ \nabla \phi(x) \, u^\varepsilon(x) \rightarrow \nabla \phi(x) \, u(x) \quad \text{in} \quad \mathbb{L}^2(\Omega)^N, \]

and hence we can handle the second term also. This means that we can let
\( \varepsilon \) tend to zero in (61) and obtain
\[
\int \sum_{k=1}^{N} \sigma_k(x) \frac{\partial x}{\partial k}(x) x_j \, dx - \int \sum_{k=1}^{N} b_{kj} \partial x_k \phi(x) \, dx = \int f(x) \phi(x) x_j \, dx.
\]
Taking (55) with \( v = \phi x_j \) into account in the right-hand side, we have
\[
\int \sum_{k=1}^{N} (\sigma_k(x) x_j - b_{kj} u(x)) \partial x_k \phi(x) \, dx = \int \sum_{k=1}^{N} \sigma_k(x) \partial x_k (\phi(x) x_j) \, dx
\]
for all \( \phi \in D(\Omega) \). Integration by parts in both sides gives
\[
\int \sum_{k=1}^{N} (-\partial x_k \sigma_k(x) x_j - \sigma_k(x) \delta_{jk} + b_{kj} \partial x_k u(x)) \phi(x) \, dx = \int \sum_{k=1}^{N} -\partial x_k \sigma_k(x) \phi(x) x_j \, dx
\]
and hence, for every \( j = 1, \ldots, N \),
\[
\int \Omega \left( \sigma_j(x) - \sum_{k=1}^{N} b_{kj} \partial x_k u(x) \right) \phi(x) \, dx = 0.
\]
Since this holds for all \( \phi \in D(\Omega) \) and \( b \) is symmetric, we conclude that
\[
\sigma(x) = b \nabla u(x).
\]
Now, since \( b \) has the necessary properties (see Proposition 4.2 i [Def] or Proposition 6.12 in [CiDo]) to provide us with a unique solution to (50), the entire sequence \( \{u^\varepsilon\} \) converges and hence we are done.

**Remark 35** The reason that the troublesome terms canceled out in (61) is that the operator is symmetric. In the non-symmetric case the adjoint operator of the local problem is used to obtain the corresponding cancellation. For details, see e.g. [CiDo].

This method may also be used for possibly nonlinear monotone problems. However, in this case the procedure of subtracting the two problems with the crosswise choice of test functions, in order to be able to pass to the limit, is not applicable. This means that one needs an alternative way to handle the limit process in the more general setting. The key to this is so-called compensated compactness, which was first introduced by Murat and Tartar; see e.g. [Mur2] and [Tar2].
Compensated compactness makes it possible to, in a certain sense, obtain convergence of the products of two sequences \( \{ u^h \} \) and \( \{ v^h \} \) to the product of the respective limits, without any of the sequences being strongly convergent. Instead, suitable convergence properties can be imposed on the derivatives in both sequences. A prototype example of compensated compactness is the following result, which is known as the div-curl lemma.

**Lemma 36** Let \( \{ u^h \} \) and \( \{ v^h \} \) be two sequences in \( L^2(\Omega)^N \) such that

\[
\begin{align*}
   u^h(x) &\to u(x) \quad \text{in} \ L^2(\Omega)^N, \\
v^h(x) &\to v(x) \quad \text{in} \ L^2(\Omega)^N.
\end{align*}
\]

Define the curl operator by

\[
\nabla \times v = \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)_{1 \leq i,j \leq N}
\]

and assume that

\[
\begin{align*}
   \nabla \cdot u^h(x) &\to \nabla \cdot u(x) \quad \text{in} \ H^{-1}(\Omega), \\
\nabla \times v^h(x) &\to \nabla \times v(x) \quad \text{in} \ L^2(\Omega)^{N \times N}.
\end{align*}
\]

Then

\[
\begin{align*}
   \int_\Omega u^h(x) \cdot v^h(x) \phi(x) \, dx \to \int_\Omega u(x) \cdot v(x) \phi(x) \, dx
\end{align*}
\]

for every \( \phi \in D(\Omega) \).

We will briefly look at the monotone correspondence to the model problem to see how this concept can be applied. More precisely, we will use the lemma below, which is a certain type of compensated compactness; see [MuTa].

**Lemma 37** Let \( \{ g^h \} \) and \( \{ u^h \} \) be two sequences in \( L^2(\Omega)^N \) and \( H^1(\Omega) \), respectively, such that

\[
\begin{align*}
   g^h(x) &\to g(x) \quad \text{in} \ L^2(\Omega)^N, \\
u^h(x) &\to u(x) \quad \text{in} \ H^1(\Omega)
\end{align*}
\]

and

\[
\nabla \cdot g^h(x) \to \nabla \cdot g(x) \quad \text{in} \ H^{-1}(\Omega).
\]

Then it holds that

\[
\begin{align*}
   \int_\Omega g^h(x) \cdot \nabla u^h(x) \phi(x) \, dx \to \int_\Omega g(x) \cdot \nabla u(x) \phi(x) \, dx
\end{align*}
\]

for every \( \phi \in D(\Omega) \).

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Let us now consider the problem
\[-\nabla \cdot a \left( \frac{x}{\varepsilon}, \nabla u^\varepsilon \right) = f(x) \quad \text{in } \Omega,
\]
\[u^\varepsilon(x) = 0 \quad \text{on } \partial \Omega,
\]
where \(a \in \mathcal{N}(\alpha, \beta, \mathbb{R}^N)\) is \(Y\)-periodic in the first argument and \(f \in H^{-1}(\Omega)\). By Theorem 21, \(G\)-convergence up to a subsequence is at hand. Furthermore, we are led by asymptotic expansion to a homogenized operator such that
\[b(\nabla u) = \int_Y a(y, \nabla u + \nabla_y u_1) \, dy \quad (62)
\]
and a local problem of the form
\[-\nabla_y \cdot a(y, \nabla u + \nabla_y u_1) = 0 \quad \text{in } \Omega \times Y. \quad (63)
\]
This is carefully carried out in [ChSm]. Using arguments similar to those used in the linear case one can show that there are functions \(u\) and \(\sigma\) such that, up to a subsequence,
\[u^\varepsilon(x) \rightharpoonup u(x) \quad \text{in } H^1_0(\Omega) \quad (64)
\]
and
\[a \left( \frac{x}{\varepsilon}, \nabla u^\varepsilon \right) \rightharpoonup \sigma(x) \quad \text{in } L^2(\Omega)^N. \quad (65)
\]
Hence, we need to prove that
\[\sigma(x) = b(\nabla u(x))
\]
with \(b\) given by (62).

As in the linear case, let us first note that the local problem has a unique solution, see e.g. [Def], Remarks 5.2 and 5.4. Now let \(u_{1,\eta} \in H^1_2(Y)/\mathbb{R}\) be the solution to (63) for a fixed \(\eta \in \mathbb{R}^N\) in the place of \(\nabla u\). If we let \(u_{1,\eta}\) still denote the periodic extension of this function to \(\mathbb{R}^N\), then by Lemma 33 we have that \(u_{1,\eta} \in H^1_{\text{loc}}(\mathbb{R}^N)\). Using Lemma 34, as in the linear case, we have
\[\int_\Omega a \left( \frac{x}{\varepsilon}, \eta + \nabla_y u_{1,\eta} \left( \frac{x}{\varepsilon} \right) \right) \cdot \nabla v(x) \, dx = 0 \quad (66)
\]
for all \(v \in H^1_0(\Omega)\). Let us now define test functions
\[w^\varepsilon_\eta(x) = \eta \cdot x + \varepsilon u_{1,\eta} \left( \frac{x}{\varepsilon} \right)
\]
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in a way similar to the above. Obviously,
\[ u_\eta^\varepsilon(x) \to \eta \cdot x \quad \text{in } H^1(\Omega) \] (67)
and from the periodicity of \( u_1, \eta \) and of \( a, w \), we get
\[ \nabla u_\eta^\varepsilon(x) \to \eta \quad \text{in } L^2(\Omega)^N \] (68)
and
\[ a \left( \frac{x}{\varepsilon}, \nabla u_\eta^\varepsilon \right) \to b(\eta) \quad \text{in } L^2(\Omega)^N. \] (69)
The monotonicity of \( a \) (see (N4) in Section 3.1.2) gives
\[
\int_{\Omega} \left( a \left( \frac{x}{\varepsilon}, \nabla u^\varepsilon \right) \right) \cdot \left( \nabla u^\varepsilon(x) - \nabla w_\eta^\varepsilon(x) \right) \phi(x) \, dx \geq 0
\]
for all \( \phi \in D(\Omega), \phi \geq 0 \). We recall that
\[ -\nabla \cdot a \left( \frac{x}{\varepsilon}, \nabla u^\varepsilon \right) = f(x) \quad \text{in } \Omega \]
and (see (66))
\[ -\nabla \cdot a \left( \frac{x}{\varepsilon}, \nabla w_\eta^\varepsilon \right) = 0 \quad \text{in } \Omega. \]
Hence \( \{-\nabla \cdot a \left( \frac{x}{\varepsilon}, \nabla u^\varepsilon \right)\} \) and \( \{-\nabla \cdot a \left( \frac{x}{\varepsilon}, \nabla w_\eta^\varepsilon \right)\} \) converge strongly in \( H^{-1}(\Omega) \) to \( f \) and zero, respectively. Since we also have the convergence properties (64)-(69) above, we can now apply the compensated compactness Lemma 37 to (70), which results in
\[
\int_{\Omega} (\sigma(x) - b(\eta)) \cdot (\nabla u(x) - \eta) \phi(x) \, dx \geq 0
\]
for all \( \phi \in D(\Omega), \phi \geq 0 \). This means that
\[ (\sigma(x) - b(\eta)) \cdot (\nabla u(x) - \eta) \geq 0 \]
for every \( \eta \in \mathbb{R}^N \) and a.e. \( x \in \Omega \). From the properties of \( a \), one can show (see Remark 5.4 in [Def]) that \( b : \mathbb{R}^N \to \mathbb{R}^N \) is monotone and hemicontinuous, and hence, by Proposition 2, maximal monotone. Thus, we have
\[ \sigma(x) = b(\nabla u(x)). \]
Also, the function \( b \) is strictly monotone (see [Def], Proposition 5.5), which means that the limit problem has a unique solution. Since this result does not depend on the choice of convergent subsequence we have accomplished what we set out to do.
The homogenization methods described in this and the previous section, i.e. the method of asymptotic expansions and Tartar’s method of oscillating test functions, can also be used to study parabolic homogenization problems. However, once the result for the elliptic case is available, there is sometimes a shortcut to the parabolic case via the comparison result in Theorem 28 in Section 3.2.2. In [Sva2], this theorem is combined with the homogenization result above to obtain a corresponding result for a monotone parabolic problem with oscillations on one microscopic scale. This approach will also be used in the next section for a parabolic problem with two different microscales.

Remark 38 In this section we have considered Tartar’s method from the periodic homogenization point of view. Let us point out that this method is quite a general one and can be applied to problems with no periodicity assumptions. It can be used, for example, to prove $G$-convergence compactness for the set $\mathcal{M}(\alpha, \beta, \Omega)$; see [MuTa]. Although there is of course no local problem that can be employed in the general case, the main idea remains the same, i.e. to choose test functions with certain convergence properties in order to be able to pass to the limit. However, the construction of the test functions is done in a more abstract way in the sense that the operators $T^\varepsilon : H^1_0(\Omega) \to H^{-1}(\Omega)$ used to define them are not explicitly given, but proven to exist and to have certain key properties. This also means that in the non-periodic case, this method does not provide any way to actually determine the limit operator.

4.3 Reiterated homogenization

So far, we have studied homogenization of problems that are characterized by two different length scales, i.e. the global scale and one microscopic scale. However, these procedures can be extended to cases in which there are more than two scale levels.

Also, when dealing with problems from the real world there is nothing to suggest that there should be a restriction to merely two scales. The heterogeneities in a material can of course occur on several different microscopic levels. This is, for example, the case if in a fiber composite there is a scale of even thinner fibers inside the fibers, or if a composite is built up of a matrix material with inclusions of different sizes and different periodicities. Figure 11 illustrates the build-up of the material in these two cases.
Homogenization problems of this kind have become known as reiterated homogenization problems, and was first studied by means of asymptotic expansions by Bensoussan et al. in the case of two microscopic scales. Here, the asymptotic expansion contains two local variables, see [BLP], Section 8.3.

A homogenization result similar to that in Section 4.2, which is also proven using the same methods, is given in [LLPW] for a monotone elliptic case with two micro-scales. Here, we reproduce a version of this result, which we will use to study a similar parabolic case.

**Theorem 39** Consider the problem

\[
-\nabla \cdot a \left( x \frac{x}{\varepsilon}, x \frac{x}{\varepsilon^2}, \nabla u^\varepsilon \right) = f(x) \quad \text{in } \Omega, \tag{71}
\]

\[
u^\varepsilon(x) = 0 \quad \text{on } \partial \Omega,
\]

where we assume that \( f \in H^{-1}(\Omega) \) and that \( a \in N(\alpha, \beta, k, \mathbb{R}^{2N}) \) is \( Y_1 \)-periodic in the first variable and \( Y_2 \)-periodic in the second variable. Furthermore, let \( \omega : \mathbb{R} \to \mathbb{R} \) be an increasing continuous function such that \( \omega(0) = 0 \) and assume that

\[
|a(y_1, y_2, \xi) - a(y_1', y_2, \xi)|^2 \leq \omega(|y_1 - y_1'|)(1 + |\xi|^2)
\]

for all \( y_1, y_1' \in Y_1 \), a.e. \( y_2 \in \mathbb{R}^N \) and all \( \xi \in \mathbb{R}^N \). Then, for a sequence \( \{u^\varepsilon\} \) of solutions in \( H^1_0(\Omega) \) to (71), it holds that

\[
u^\varepsilon(x) \rightharpoonup u(x) \quad \text{in } H^1_0(\Omega)
\]

where \( u \in H^1_0(\Omega) \) is the unique solution to the problem

\[
-\nabla \cdot b(\nabla u) = f(x) \quad \text{in } \Omega,
\]

\[
u(x) = 0 \quad \text{on } \partial \Omega.
\]
Here, \( b \) is defined by
\[
b(\nabla u) = \int_{Y^2} a(y_1, y_2, \nabla u + \nabla y_1 u_1 + \nabla y_2 u_2) \, dy_2 dy_1,
\]
where \( u_1 \in L^2(\Omega; H^1_\#(Y_1)/\mathbb{R}) \) and \( u_2 \in L^2(\Omega \times Y_1; H^1_\#(Y_2)/\mathbb{R}) \) uniquely solve the local problems
\[
- \nabla y_2 \cdot a(y_1, y_2, \nabla u + \nabla y_1 u_1 + \nabla y_2 u_2) = 0, \\
- \nabla y_1 \cdot \int_{Y_2} a(y_1, y_2, \nabla u + \nabla y_1 u_1 + \nabla y_2 u_2) \, dy_2 = 0.
\]

**Proof.** See Theorem 3.1 in [LLPW]. ■

**Remark 40** A generalization of this result to operators of the form
\[- \nabla \cdot a(x, \xi, \xi, \nabla(\cdot))\] is given in Theorem 4.1 in [LLPW], where multiscale convergence techniques are used.

Let us now investigate a parabolic equation that also has two scales on the microscopic level and where \( a \) is time-dependent. We consider the problem
\[
\partial_t u^\varepsilon(x,t) - \nabla \cdot a \left( t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \nabla u^\varepsilon \right) = f(x,t) \quad \text{in } \Omega_T, \\
u^\varepsilon(x,0) = u^0(x) \quad \text{in } \Omega, \\
u^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T),
\]
where we let \( u^0 \in L^2(\Omega) \) and \( f \in L^2(0,T; H^{-1}(\Omega)) \). We will prove a homogenization result for this equation, benefiting from the result above concerning the elliptic problem.

We assume that
\[
a : \mathbb{R}_+ \times \mathbb{R}^{3N} \to \mathbb{R}^N
\]
satisfies the following conditions, where \( 0 < k \leq 1 \) and \( \alpha \) and \( \beta \) are positive constants:

(i) \( a(t, y_1, y_2, 0) = 0 \) for all \( (t, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}^{2N} \).

(ii) \( a(\cdot, \cdot, \cdot, \xi) \) is \( Y^2 \)-periodic in \( (y_1, y_2) \) and continuous for all \( \xi \in \mathbb{R}^N \).

(iii) \( a(t, y_1, y_2, \cdot) \) is continuous for all \( (t, y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}^{2N} \).
Theorem 41

Consider the problem (72). It holds that

(a) the problem has a unique solution \( u^\varepsilon \in H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega)) \) for every fixed \( \varepsilon > 0 \).

(b) the solutions \( u^\varepsilon \) are uniformly bounded in \( L^\infty(0,T;L^2(\Omega)) \) and \( H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega)) \), i.e., for some positive constant \( C \) it holds that

\[
\|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C
\]

and

\[
\|u^\varepsilon\|_{H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega))} \leq C.
\]

(c) the sequence of functions

\[
a^h(x,t,\xi) = a\left(t,\frac{x}{\varepsilon},\frac{x}{\varepsilon^2},\xi\right)
\]

where \( \varepsilon = \varepsilon(h) \to 0 \) as \( h \to \infty \) and \( (x,t,\xi) \in \Omega_T \times \mathbb{R}^N \) belongs to \( \mathcal{N}(\alpha,\beta,k,\Omega_T) \). Hence \( \{a^h\} \) \( G \)-converges up to a subsequence, to a limit \( b \in \mathcal{N}(\alpha,\beta,\bar{k},\Omega_T) \), where \( \bar{k} = k/(2-k) \) and \( \bar{\beta} \) is a positive constant depending on \( \alpha, \beta \) and \( k \) only.
Proof. For (a) we refer to Theorem 8 in Section 2.2.2, and for the first a priori estimate in (b) to Lemma 30.3 in [ZeiIII]. Let us now define

$$a^h(x, t, \xi) = a \left( t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}; \xi \right)$$  \hspace{1cm} (73)$$

where \((x, t, \xi) \in \Omega_T \times \mathbb{R}^N\) and with \(\varepsilon > 0\), where \(\varepsilon = \varepsilon(h) \to 0\) as \(h \to \infty\). We will prove that this function belongs to \(\mathcal{N}(\alpha, \beta, k, \Omega_T)\), i.e., that it fulfills the requirements \((N1) - (N4)\) in Section 3.2.2.

From assumption \((i)\) we immediately get

$$a \left( t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, 0 \right) = 0,$$

which means that condition \((N1)\) is satisfied.

We have that \(a(\cdot, \cdot, \cdot, \xi)\) is measurable for every fixed \(\xi \in \mathbb{R}^N\) because \(a(\cdot, \cdot, \cdot, \xi)\) is continuous for every such \(\xi\) by \((ii)\). Hence, condition \((N2)\) is fulfilled.

Furthermore, from the fact that \(a\) satisfies \((iv)\) it follows that

$$\left| a \left( t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \xi \right) - a \left( t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \xi' \right) \right| \leq \beta(1 + |\xi| + |\xi'|^{1-k}) |\xi - \xi'|^k$$

holds for all \((x, t) \in \Omega_T\), all \(\xi, \xi' \in \mathbb{R}^N\) and \(0 < k \leq 1\). This means that \(a\) agrees with condition \((N3)\).

Finally, condition \((N4)\) is satisfied because

$$\left( a \left( t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \xi \right) - a \left( t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \xi' \right) \right) \cdot (\xi - \xi') \geq \alpha |\xi - \xi'|^2$$

for all \((x, t) \in \Omega_T\) and all \(\xi, \xi' \in \mathbb{R}^N\) since \(a\) fulfills \((v)\). Hence, \(a^h\) given by (73) belongs to \(\mathcal{N}(\alpha, \beta, k, \Omega_T)\). By Proposition 5.3 and Corollary 5.1 in [Sva1], we can then deduce that the solutions \(u^\varepsilon\) and the derivatives \(\partial_t u^\varepsilon\) are bounded in \(L^2(0, T; H^1_0(\Omega))\) and \(L^2(0, T; H^{-1}(\Omega))\), respectively, and hence the second a priori estimate follows. Also, Theorem 27, on parabolic \(G\)-convergence up to a subsequence, applies and the proof is complete. \(\blacksquare\)

Remark 42 Note that since we have proven that the conditions \((N1)-(N4)\) in Section 3.2.2 are satisfied for all \(t \in (0, T)\), we know, apart from the fact that \(a^h\) is in \(\mathcal{N}(\alpha, \beta, k, \Omega_T)\), that it belongs to \(\mathcal{N}(\alpha, \beta, k, \Omega)\) for every fixed \(t\); see the corresponding conditions in Section 3.1.2.
Thus, we know that we have a limit problem of the desired form; that is, a problem of the same type as (72) with the governing function \( b \) having the properties necessary to imply a unique solution, for example. Now we will characterize the limit function \( b \) further by proving the homogenization result below. The proof will be done by utilizing the conclusion about G-convergence from Theorem 41 and the homogenization result for elliptic reiterated problems in Theorem 39, together with the comparison result for G-limits in Theorem 28.

**Theorem 43** Let \( \{ u^\varepsilon \} \) be a sequence of solutions in \( H^1 (0, T; H^1_0 (\Omega), H^{-1} (\Omega)) \) to (72). Then it holds that

\[
\begin{align*}
u^\varepsilon (x, t) & \to u(x, t) \quad \text{in} \ L^2 (0, T; H^1_0 (\Omega)) \\
u^\varepsilon (x, t) & \to u(x, t) \quad \text{in} \ L^2 (\Omega_T)
\end{align*}
\]

where \( u \in H^1 (0, T; H^1_0 (\Omega), H^{-1} (\Omega)) \) is the unique solution to the homogenized problem

\[
\begin{align*}
\partial_t u(x, t) - \nabla \cdot b(t, \nabla u) &= f(x, t) \quad \text{in} \ \Omega_T, \\
u(x, 0) &= u^0(x) \quad \text{in} \ \Omega, \\
u(x, t) &= 0 \quad \text{on} \ \partial \Omega \times (0, T).
\end{align*}
\]

Here, \( b \) is given by

\[
b(t, \nabla u) = \int_{Y^2} a(t, y_1, y_2, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) \ dy_2 dy_1,
\]

where \( u_1(t) \in L^2 (\Omega; H^1_1 (Y_1) / \mathbb{R}) \) and \( u_2(t) \in L^2 (\Omega \times Y_1; H^1_1 (Y_2) / \mathbb{R}) \) uniquely solve the local problems

\[
\begin{align*}
-\nabla_{y_2} \cdot a(t, y_1, y_2, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) &= 0, \\
-\nabla_{y_1} \cdot \int_{Y_2} a(t, y_1, y_2, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) \ dy_2 &= 0.
\end{align*}
\]

**Proof.** So far, we know—by Theorem 41—that we have G-convergence, at least for a subsequence, of the operators corresponding to (72), i.e. for the sequence \( \{ a^h \} \) given by

\[
a^h(x, t, \xi) = a \left( t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \xi \right),
\]

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where \( \varepsilon = \varepsilon(h) \to 0 \) as \( h \to \infty \), we have \( G \)-convergence in the sense of parabolic operators to some \( b \), up to a subsequence. To apply the comparison result in Theorem 28, we must also prove that the condition

\[
|a^h(x, t, \xi) - a^h(x, t', \xi)| \leq B(t - t')(1 + |\xi|),
\]

where \( B : \mathbb{R}_+ \to \mathbb{R}_+ \) is an increasing continuous function such that \( B(t) \to 0 \) as \( t \to 0_+ \), is satisfied for all \( \xi \in \mathbb{R}^N \), a.e. \( x \in \Omega \) and all \( t \) and \( t' \) such that \( 0 < t' < t < T \). Since \( a \) satisfies \( (vi) \) it holds that

\[
|a \left( t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \xi \right) - a \left( t', \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \xi \right) | \leq B(t - t')(1 + |\xi|)
\]

for all \( x \in \Omega \), all \( \xi \in \mathbb{R}^N \) and \( t, t' \) such that \( 0 < t' < t < T \). Hence (75) is fulfilled.

Finally, we need to know that we have \( G \)-convergence in the sense of elliptic operators for \( t \) fixed. This is provided by Theorem 29 in Section 3.2. We have already proven that the conditions stated for \( \{a^h\} \) in Theorem 29 are satisfied; that is, (75) is fulfilled and \( a^h(t) \) belongs to \( \mathcal{N}(\alpha, \beta, k, \Omega) \) for every fixed \( t \); see Remark 42. Thus, we have \( G \)-convergence of \( \{a^h(t)\} \) in the sense of elliptic operators to a function \( b^r(t) \), at least for a subsequence, and with the same subsequence for all \( t \in (0, T) \).

Now we know that the requirements needed to apply Theorem 28 are all fulfilled. Hence \( b = b^r \) and since all assumptions made in Theorem 39 concerning the elliptic case are satisfied, we can employ this to determine \( b^r(t) \) for any fixed \( t \). Thus the desired result follows, at least for a subsequence, except for the convergence (74). This convergence property is a consequence of the second a priori estimate in \( (b) \) in Theorem 41; see e.g. Problem 23.13b in [ZeiIIA]. Since the same result will follow for any \( G \)-convergent subsequence, the entire sequence \( G \)-converges and the proof is complete.

\[ \text{Remark 44} \]

Parabolic homogenization problems similar to the one treated in this section will be studied in Section 5.3. In these problems, there are oscillations also in the time variable and the homogenization results will be proven by multiscale convergence techniques.
5 Homogenization by multiscale convergence techniques

In this chapter we will investigate an alternative method for periodic homogenization. A technique based on so-called two-scale convergence has proven to be efficient for problems with one microscopic scale, such as the model problem studied in Chapters 1 and 4. As in Tartar’s method, the choice of test functions plays an essential role, but in contrast to that approach, this choice does not rely on the local problem. This means that we are not dependent on the method of asymptotic expansions to find the local problem beforehand. Instead, the local and the homogenized problem are immediately obtained in the coupled form, after passing to the limit for certain choices of test functions.

In Section 5.1 we recall some fundamentals about two-scale convergence and show how it can be used to perform the homogenization of the model problem. We also investigate the connection between a mode of convergence of two-scale type and the asymptotic expansion. Two-scale convergence and the homogenization method based on this have been generalized to multiple scales and used to solve corresponding reiterated homogenization problems. In Section 5.2, we illustrate this for an elliptic problem with two micro-scales. We then extend this approach in Section 5.3 to fit problems that may have periodic oscillations in time as well. In particular we prove a compactness result for gradients adapted to a problem with two microscopic scales in space and one fast scale in time, such as the parabolic problem (11) with $a^\varepsilon$ given by (13). We then proceed with the homogenization of this problem. Finally, we perform the homogenization of (11) for the choice of $a^\varepsilon$ given in (14), where we have one microscopic spatial scale and two fast temporal scales.

5.1 Homogenization with two scales

In 1989, Nguetseng introduced what has come to be known as two-scale convergence. A significant property of this type of convergence is that while the functions in the sequence depend on one variable, $x \in \mathbb{R}^N$ only, the limit function contains two variables. As we will see, this means that it captures oscillations on a microscopic scale, which are averaged away in usual weak limits. This explains the name, and is also what makes two-scale convergence particularly suitable for periodic homogenization. After studying
two-scale convergence with respect to some properties of the test functions and its relationship to other modes of convergence in Section 5.1.1 we apply it to homogenization in Section 5.1.2. Finally, we discuss the relationship between two-scale convergence and asymptotic expansions in Section 5.1.3.

5.1.1 Two-scale convergence

In [Ngu1], Nguetseng presented a result on compactness with respect to a convergence of two-scale type. He proved that for every bounded sequence \( \{ u_\varepsilon \} \) in \( L^2(\Omega) \), there is a function \( u_0 \) in \( L^2(\Omega \times \mathbb{Y}) \) such that, up to a subsequence,

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon(x) v\left(x, \frac{x}{\varepsilon}\right) \, dx = \int_{\Omega} \int_{\mathbb{Y}} u_0(x, y) v(x, y) \, dy \, dx
\]

for all \( v \in C_0(\overline{\Omega}, C_0(\mathbb{Y})) \) and all \( v = v_1v_2, \ v_1 \in C_0(\overline{\Omega}), \ v_2 \in L^2(\mathbb{Y}) \) when \( \Omega \) is bounded, and for all \( v \in C_0(\overline{\Omega}, C_0(\mathbb{Y})) \) and all \( v = v_1v_2, \ v_1 \in C_0(\overline{\Omega}), \ v_2 \in L^2(\mathbb{Y}) \) when \( \Omega \) is equal to \( \mathbb{R}^N \).

Allaire extended this result to hold for some less restricted classes of test functions, and the proof is also significantly different; see [All1]. One of these spaces of test functions is \( L^2(\Omega; C_0(\mathbb{Y})) \), and this class has been chosen in the exact contemporary definition of two-scale convergence.

**Definition 45** A sequence \( \{ u_\varepsilon \} \) in \( L^2(\Omega) \) is said to two-scale converge to a limit \( u_0 \in L^2(\Omega \times \mathbb{Y}) \), called the two-scale limit, if

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u_\varepsilon(x) v\left(x, \frac{x}{\varepsilon}\right) \, dx = \int_{\Omega} \int_{\mathbb{Y}} u_0(x, y) v(x, y) \, dy \, dx
\]

for all \( v \in L^2(\Omega; C_0(\mathbb{Y})) \). This is written

\[
u_\varepsilon(x) \rightharpoonup u_0(x, y).
\]

We have the following theorem on compactness with respect to two-scale convergence.

**Theorem 46** A sequence \( \{ u_\varepsilon \} \) bounded in \( L^2(\Omega) \) has a two-scale convergent subsequence.

**Proof.** See the proof of Theorem 1.2 in [All1]. ■

To see what a two-scale limit might look like, we study the sequence \( \{ u_\varepsilon \} \) given by

\[
u_\varepsilon(x) = \ln\left(x + \sin\frac{2\pi x}{\varepsilon}\right).
\]
This sequence is weakly convergent in $L^2(\Omega)$ e.g. for $\Omega = (2, 3)$. We obtain the following appearance of $u^\varepsilon$ for some different values of $\varepsilon$.

![Figure 12. $u^\varepsilon$ for some values of $\varepsilon$.](image)

To gain a better understanding of the two-scale limit and its connection to the weak $L^2(\Omega)$-limit, we first recall that weak convergence of $\{u^\varepsilon\}$ to $u$ in $L^2(\Omega)$ means that

$$
\int_{\Omega} u^\varepsilon(x)v(x) \, dx \to \int_{\Omega} u(x)v(x) \, dx
$$

for all $v \in L^2(\Omega)$. For $v = 1$, this simply means that the sequence of mean values converges—or the sequence of areas under the graph for our choice of $u^\varepsilon$. Since this must also hold, for example, for functions $v$ that are equal to one on any small interval of $\Omega$ and zero elsewhere, the convergence of the mean values of $u^\varepsilon$ will also take place locally and hence the weak limit $u$ follows the trend of the functions in the sequence, as in Figure 13.

![Figure 13. The weak $L^2(\Omega)$-limit of $\{u^\varepsilon\}$.](image)

Thus, this limit of $\{u^\varepsilon\}$ could be understood as a way to represent the limit of the mean value locally, which implies that it does not contain any information about what the rapid oscillations look like.
For two-scale convergence, we require that
\[
\int_{\Omega} u^\varepsilon(x) v\left(x, \frac{x}{\varepsilon}\right) \, dx \to \int_{\Omega} \int_{Y} u_0(x, y) v(x, y) \, dy \, dx
\] 
for all \( v \in L^2(\Omega; C^1_0(Y)) \). Again, choosing \( v = 1 \), this obviously means that the areas under the graphs of the functions in \( \{u^\varepsilon\} \) approach the volume under the graph of the function \( u_0 \), or, equivalently, that the sequence of mean values of \( u^\varepsilon \) converges to the mean value of \( u_0 \). Thus, our first estimate of the two-scale limit might be a plane surface with height equal to the limit of the mean values of \( \{u^\varepsilon\} \).

Figure 14. A first estimate of the two-scale limit.

In a way similar to that described above, we conclude that this convergence will take place locally in \( \Omega \); i.e. in any small interval, the area under the graph of \( u^\varepsilon \) approaches the volume of the corresponding slice under the graph of \( u_0 \). This means that \( u_0 \) will be different at different locations in \( \Omega \).

Figure 15. Trend in the \( x \)-direction.
A naive guess at the two-scale limit for \( \{ u^\varepsilon \} \) in (77) might then be the function above. This suggestion has the essential feature that the mean value of \( u_0 \) over \( Y \) coincides with the weak \( L^2(\Omega) \)-limit \( u \); see (84). It remains for us to find out how the two-scale limit is actually shaped in the \( y \)-direction.

Our choice of \( \{ u^\varepsilon \} \) contains oscillations of the same frequency as \( v(x, \frac{x}{\varepsilon}) \). This is reflected in the fact that \( u_0 \) varies in the \( y \) variable for \( x \) fixed. This is perhaps most easily realized if we study \( Y \)-periodic test functions that are equal to one on some small fraction of the period and zero elsewhere. To capture the local variations together with the global tendency observed above, we choose \( v \) as the product between such a function and a cutoff function, whose support is contained in a small interval in \( \Omega \).

![Figure 16. \( v(x, \frac{x}{\varepsilon}) \) for two choices of test functions \( v \).](image)

For this kind of test functions, (78) means that the area under the graph of \( u^\varepsilon \) on the small intervals where the function \( v(x, \frac{x}{\varepsilon}) \) is equal to one approaches the volume under the graph of \( u_0 \) on the small rectangle in \( \Omega \times Y \) where the test function \( v \) has support.

![Figure 17. The product \( u^\varepsilon(x)v(x, \frac{x}{\varepsilon}) \) for \( v(x, \frac{x}{\varepsilon}) \) as above.](image)

Thus, the mean height of \( u_0 \) on such a rectangle is determined by the height of the bars under the graph of \( u^\varepsilon \), see Figure 17. In Figure 18, the two leftmost bars correspond to the two choices of test functions above. Obviously, the height differs depending on where in the period \( v \) equals one.
We also get different heights for different choices of small intervals in $\Omega$ depending on the global tendency of $u^\varepsilon$, and duly we recognize the trend in the $x$-direction from Figure 15 in the picture to the right in Figure 18. We end up with the two-scale limit $u_0$ displayed below.

The trend in the $x$-direction seems to be in accordance with the weak limit, but in the two-scale limit we also see that the local fluctuations are reflected in oscillations in the $y$-direction, which is typical for sequences $\{u^\varepsilon\}$ of oscillating functions of the kind studied above. However, two-scale convergence is not restricted to sequences of this type. Any function $u_0 \in L^2(\Omega \times Y)$ can be obtained as a two-scale limit to some bounded sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ (see [All1], Lemma 1.13), and we already know that any such sequence has a two-scale convergent subsequence. The character of the $y$-dependence in $u_0$ depends on whether $u^\varepsilon$ contains oscillations that the oscillations in $v(x, \frac{x}{\varepsilon})$ can capture.
The choice of test functions is crucial in order to obtain a useful concept of convergence. Firstly, we need to require that $v$ is periodic in the $y$ variable. We must also be able to substitute $\frac{x}{\varepsilon}$ for $y$ and still get a measurable function. This means that the test functions $v$ must satisfy the following:

$$v(\cdot, y) \text{ is measurable for all } y \in Y,$$

$$v(x, \cdot) \text{ is continuous for a.e. } x \in \Omega.$$  

These conditions are known as Carathéodory conditions and allow us to replace $y$ with a measurable function of $x$ and still obtain a measurable function; see p. 1013 in the appendix in [ZeiII].

Within the scope of these assumptions, it is possible to define different classes of test functions for which compactness results of the kind given in Theorem 46 hold true. Examples of such spaces of test functions are $L^2(\Omega; C^1(\bar{Y}))$, $L^2(\bar{Y}; C(\Omega))$ and $C(\bar{\Omega}; C^1(\bar{Y}))$ for $\Omega$ bounded, and $L^2(\Omega; C^1(\bar{Y}))$ when $\Omega$ may be unbounded, e.g. when $\Omega$ is equal to $\mathbb{R}^N$; see [All1] and [All2].

The proof is based on some properties possessed by these spaces of so-called admissible test functions. If we denote any of these classes by $B(\Omega, Y)$, $B(\Omega, Y)$ is a separable Banach space that is dense in $L^2(\Omega \times Y)$. We also have for all $v \in B(\Omega, Y)$

$$\left\| v \left( x, \frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega)} \leq \| v(x, y) \|_{B(\Omega, Y)}$$  \hspace{1cm} (79)

and

$$\left\| v \left( x, \frac{x}{\varepsilon} \right) \right\|_{L^2(\Omega)} \to \| v(x, y) \|_{L^2(\Omega \times Y)}.$$  \hspace{1cm} (80)

Let us summarize the proof given in [All1]. Consider the integral in the left-hand side of (76) as a linear functional $F^\varepsilon$ acting on $v \in B(\Omega, Y)$. By Hölder’s inequality and (79), $F^\varepsilon$ is bounded with respect to the $B(\Omega, Y)$-norm and thus belongs to $(B(\Omega, Y))'$. For $\{u^\varepsilon\}$ bounded in $L^2(\Omega)$, the sequence of functionals

$$\langle F^\varepsilon, v \rangle_{(B(\Omega, Y))', B(\Omega, Y)} = \int_\Omega u^\varepsilon(x) v \left( x, \frac{x}{\varepsilon} \right) dx$$

is bounded in $(B(\Omega, Y))'$, and since $B(\Omega, Y)$ is separable $\{F^\varepsilon\}$ has a weak* limit $F \in (B(\Omega, Y))'$, up to a subsequence. Finally, from (80) it follows that $F$ is bounded also with respect to the $L^2(\Omega \times Y)$-norm, and thus by density we can extend $F$ to a functional $G \in (L^2(\Omega \times Y))'$. Hence, by Riesz
representation theorem, we deduce that there is a unique $u_0 \in L^2(\Omega \times Y)$ such that

$$\langle G, v \rangle_{(L^2(\Omega \times Y)', L^2(\Omega \times Y))} = \int_{\Omega} \int_Y u_0(x, y) v(x, y) \, dy \, dx.$$ 

We will now investigate the class $L^2(\Omega; C^1_\varepsilon(Y))$ of admissible test functions used in the definition in more detail. Some important properties of these functions, such as (80), are based on the fact that for any function $f \in L^1(\Omega; C^1_\varepsilon(Y))$ it holds that

$$\lim_{\varepsilon \to 0} \int_{\Omega} f \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_Y f(x, y) \, dy \, dx; \quad (81)$$

see e.g. Theorem 2 in [LNW]. This implies that for $v \in L^2(\Omega; C^1_\varepsilon(Y))$, and hence $v^2 \in L^1(\Omega; C^1_\varepsilon(Y))$, we have

$$\lim_{\varepsilon \to 0} \int_{\Omega} v^2 \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_Y v^2(x, y) \, dy \, dx$$

and obviously (80) follows. Also, if we let $w \in L^2(\Omega)$, we have—still for $v \in L^2(\Omega; C^1_\varepsilon(Y))$—that $vw \in L^1(\Omega; C^1_\varepsilon(Y))$ and thus we obtain from (81) that

$$\lim_{\varepsilon \to 0} \int_{\Omega} v \left( x, \frac{x}{\varepsilon} \right) w(x) \, dx = \int_{\Omega} \int_Y v(x, y) w(x) \, dy \, dx$$

for any $w \in L^2(\Omega)$, which means that

$$v \left( x, \frac{x}{\varepsilon} \right) \rightharpoonup \int_Y v(x, y) \, dy \quad \text{in} \ L^2(\Omega). \quad (82)$$

We will see that this allows a link between the two-scale limit and the strong limit. 

Let us now consider a couple of results relating two-scale convergence to strong and weak convergence in $L^2(\Omega)$, respectively. If

$$u^\varepsilon(x) \to u(x) \quad \text{in} \ L^2(\Omega),$$

the power of possible oscillations can be forced to be arbitrarily small by choosing $\varepsilon$ small enough; see Figure 5 for example. This means that for such
a sequence, there should not be any substantial local oscillations for the two-scale limit to capture, i.e., we could expect that it is constant in \( y \). Thus, it is not surprising that for a sequence strongly convergent in \( L^2(\Omega) \) we have

\[
u^\varepsilon (x) \rightharpoonup u (x);
\]

that is, a strongly convergent sequence \( \{ u^\varepsilon \} \) two-scale converges to its strong limit \( u \) and hence, for such a sequence, the second variable vanishes in the two-scale limit. To see this, we simply use (82) and get for \( v \in L^2 (\Omega; C_\infty (Y)) \)

\[
\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon (x) v \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_\Omega u(x) \int_Y v (x, y) \, dy \, dx = \int_\Omega \int_Y u(x)v (x, y) \, dy \, dx,
\]

where the passage to the limit is easily done since we have a product of one strongly and one weakly convergent sequence in \( L^2(\Omega) \).

If \( \{ u^\varepsilon \} \) does not converge strongly, we can still get a connection to the weak limit. If

\[
u^\varepsilon (x) \rightharpoonup u_0 (x, y),
\]

then

\[
u^\varepsilon (x) \rightharpoonup \int_Y u_0 (x, y) \, dy \quad \text{in } L^2(\Omega); \tag{84}
\]

that is, the weak limit of the sequence is obtained by integrating the two-scale limit over \( Y \). This follows immediately from the definition of two-scale convergence if we choose test functions independent of \( y \), i.e. in \( L^2(\Omega) \). Obviously, when deriving the weak limit from the two-scale limit the information about the local oscillations that we saw in Figure 19 is lost.

Two-scale convergence has sometimes been defined as the requirement that (76) should hold for all functions \( v \) in the set \( D(\Omega; C_\infty (Y)) \). This is, however, too restricted a choice of test functions in the sense that we lose the connection to the weak limit. Indeed, a limit in this sense can be obtained also for unbounded sequences in \( L^2(\Omega) \), and since an unbounded sequence does not possess a weak limit there is no such limit to relate to as in (84). However, for a sequence bounded in \( L^2(\Omega) \), the fact that (76) holds for every \( v \) in \( D(\Omega; C_\infty (Y)) \) guarantees that the entire sequence two-scale converges to \( u_0 \), and not just a subsequence. See also comments on this in [LNW].
When using two-scale convergence for the purpose of homogenization, we will need information about the two-scale limit of the gradient of $u^\varepsilon$. For a sequence $\{u^\varepsilon\}$ bounded in $H^1(\Omega)$, we have the characterization in the following theorem.

**Theorem 47** Let $\{u^\varepsilon\}$ be a sequence bounded in $H^1(\Omega)$. Then there exists $u \in H^1(\Omega)$ and $u_1 \in L^2(\Omega; H^1_Y(Y)/\mathbb{R})$ such that, for a subsequence, it holds that

$$u^\varepsilon(x) \rightharpoonup u(x) \quad \text{in } H^1(\Omega)$$

and

$$\nabla u^\varepsilon(x) \rightharpoonup \nabla u(x) + \nabla_y u_1(x,y).$$

**Proof.** See Theorem 3 in [Ngu1] and Proposition 1.14 in [All1]. □

Before we turn our interest to the homogenization let us point out that two-scale convergence can be defined in a more general setting, where no periodicity assumptions are made. If we consider the left-hand side in (76) to be a special case of functionals of the type

$$F^h(v) = \int_\Omega u^h(x) (\tau^h v)(x) \, dx$$

where $\{\tau^h\}$ is a sequence of maps acting on functions $v \in L^2(\Omega; C^0(Y))$, i.e.

$$\tau^h(v(x,y)) = (\tau^h v)(x) = v\left(x, \frac{x}{\varepsilon}\right),$$

where $\varepsilon = \varepsilon(h) \to 0$ as $h \to \infty$, we obtain a generalization of periodic two-scale convergence in a natural way.

Imposing certain conditions on the maps $\tau^h$ making up the key properties necessary for two-scale compactness to hold, i.e. properties corresponding to (79) and (80), it is possible to obtain a compactness result of the same kind as for periodic two-scale convergence. We assume that $\{u^h\}$ is a bounded sequence in $L^2(\Omega)$, that $X$ is a separable Banach space contained in $L^2(\Omega \times A)$, where $A$ is an open bounded subset of $\mathbb{R}^M$, and $\tau^h : X \to L^2(\Omega)$ linear maps such that the conditions

$$\|\tau^h v\|_{L^2(\Omega)} \leq C \|v\|_X,$$

$$\lim_{h \to \infty} \|\tau^h v\|_{L^2(\Omega)} \leq C \|v\|_{L^2(\Omega \times A)}$$

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are met. Then there is $u_0 \in L^2(\Omega \times A)$ such that, up to a subsequence,

$$\lim_{h \to \infty} \int_{\Omega} u^h(x)(\tau^h v)(x) \, dx = \int_{\Omega} \int_{A} u_0(x,y)v(x,y) \, dydx$$

for all $v \in X$; see e.g. [Hol1], [HSSW], [Sil1] and [HOS].

Let us point out that in order to achieve a connection to the usual weak limit, like (84), additional assumptions are needed, see [Sil1] and [Sil2].

Finally we remark that other generalizations of two-scale convergence allowing non-periodic functions have been made, such as scale convergence (see [MaTo]) and $\Sigma$-convergence (see [Ngu2]).

### 5.1.2 Homogenization using two-scale convergence

Now let us see how two-scale convergence can be applied to carry out the homogenization for the model problem

$$\begin{align*}
-\nabla \cdot \left( a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon(x) \right) &= f(x) \quad \text{in} \, \Omega, \\
u^\varepsilon(x) &= 0 \quad \text{on} \, \partial \Omega,
\end{align*}$$

where $f \in L^2(\Omega)$ and $a \in M(\alpha, \beta, \mathbb{R}^N)$ is $Y$-periodic. The point of departure is the weak form of the equation, namely that the equality

$$\int_{\Omega} a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x)v(x) \, dx$$

should hold for all $v \in H^1_0(\Omega)$. Using two different classes of test functions we will get the homogenized problem and the local problem as two coupled PDEs.

Recall that $\{u^\varepsilon\}$ is bounded in $H^1_0(\Omega)$ and hence in $H^1(\Omega)$. This means that

$$u^\varepsilon(x) \rightharpoonup u(x) \quad \text{in} \, H^1_0(\Omega)$$

up to a subsequence, and furthermore that the compactness result for the gradients in Theorem 47 is applicable. First, we choose test functions $v$ with no microscopic oscillations. All products of the components in $a$ and $\nabla v$ create admissible test functions (see Theorem 16 in [LNW]) and hence we get, after passing to the limit, the global problem

$$\int_{\Omega} \int_{Y} a(y)(\nabla u(x) + \nabla_y u_1(x,y)) \cdot \nabla v(x) \, dydx = \int_{\Omega} f(x)v(x) \, dx$$

(86)

for all $v \in H^1_0(\Omega)$. 66
If we choose test functions oscillating in time with $a \left( \frac{x}{\varepsilon} \right)$, more precisely

$$v(x) = \varepsilon v_1(x) v_2 \left( \frac{x}{\varepsilon} \right),$$

where $v_1 \in D(\Omega)$ and $v_2 \in C^\infty_\#(Y)/\mathbb{R}$, we get after differentiation of $v$

$$\int_\Omega \left( \int a \left( \frac{x}{\varepsilon} \right) \nabla u_\varepsilon(x) \cdot \left( \varepsilon \nabla v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) + v_1(x) \nabla_y v_2 \left( \frac{x}{\varepsilon} \right) \right) \right) \, dx =$$

$$\int_\Omega f(x) \varepsilon v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) \, dx.$$

When passing to the limit, the right-hand side vanishes as well as one term in the left-hand side. Using Theorem 47 for the remaining term, we have

$$\int_\Omega \int_Y a(y) (\nabla u(x) + \nabla_y u_1(x,y)) v_1(x) \nabla_y v_2(y) \, dy \, dx = 0.$$

Applying the variational lemma to eliminate the integral over $\Omega$, we end up with the local problem

$$\int_Y a(y) (\nabla u(x) + \nabla_y u_1(x,y)) \cdot \nabla_y v_2(y) \, dy = 0$$

holding for all $v_2 \in C^\infty_\#(Y)/\mathbb{R}$ and hence, by density, for all $v_2 \in H^1_\#(Y)/\mathbb{R}$. We now have a system of two PDEs and two unknowns, $u$ and $u_1$.

In some lucky cases the problems can be decoupled using separation of variables, enabling us to rewrite the local problem without the appearance of $x$. Using the ansatz

$$u_1(x,y) = \nabla u(x) \cdot z(y),$$

where $z \in (H^1_\#(Y)/\mathbb{R})^N$ in (86) we obtain the homogenized problem which means that

$$\int_\Omega b \nabla u(x) \cdot \nabla v(x) \, dx = \int_\Omega f(x) v(x) \, dx \quad (87)$$

for all $v \in H^1_0(\Omega)$, where

$$b_{ij} = \int_Y a_{ij}(y) + \sum_{k=1}^N a_{ik}(y) \partial_{y_k} z_j(y) \, dy.$$

The variable separated version of the local problem becomes

$$\int_Y a(y) (e_j + \nabla_y z_j(y)) \cdot \nabla_y v_2(y) \, dy = 0,$$

still for all $v_2 \in H^1_\#(Y)/\mathbb{R}$.

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Thus, we can determine $z$ from the local problem and use it to identify the deviation of $b$ from the arithmetic mean of $a$. Hence, $z$ tells us how $u_1$ represents the impact of the rapid oscillations in $a(\frac{x}{\varepsilon})$ on the limit coefficient. Since the characterization of $b$ is not dependent on the subsequence and the limit equation (87) has a unique solution (see e.g. Section 6.3 in [CiDo]) we have convergence of the entire sequence.

Summing up, we have, writing in classical form, that for a sequence $\{u^\varepsilon\}$ of solutions to (85) it holds that

$$u^\varepsilon(x) \rightarrow u(x) \quad \text{in} \quad H^1_0(\Omega),$$

where $u$ uniquely solves the homogenized problem

$$-\nabla \cdot (b \nabla u(x)) = f(x) \quad \text{in} \quad \Omega,$$

$$u(x) = 0 \quad \text{on} \quad \partial \Omega,$$

with

$$b_{ij} = \int_Y a_{ij}(y) + \sum_{k=1}^N a_{ik}(y) \partial_{y_k} z_j(y) \, dy.$$

Here, $z_j \in H^1_0(Y) / \mathbb{R}$ is the unique solution to

$$-\nabla_y \cdot (a(y)(e_j + \nabla_y z_j(y))) = 0 \quad \text{in} \quad Y.$$

This means that we have now proven Theorem 32 by two-scale convergence techniques.

### 5.1.3 Two-scale convergence and asymptotic expansions

In this section we investigate the connection between the asymptotic expansion discussed in Section 4.1 and two-scale convergence. In particular, we will see how a convergence of two-scale type can contribute to the interpretation of the second term in the expansion. Generalizations of these observations will turn out to be useful in the homogenization procedures in Section 5.3.

We have seen that the two-scale convergence method and the method of asymptotic expansions both provide us with the homogenization result for the model problem. To enhance the understanding of the relation between these two methods, we consider this result with the homogenized and the local problem in the coupled form, where the function $u_1$ is visible.
Let us again recall the problem
\[-\nabla \cdot \left( a \left( \frac{x}{\varepsilon} \right) \nabla u^\varepsilon (x) \right) = f (x) \quad \text{in } \Omega, \tag{88}\]
\[u^\varepsilon (x) = 0 \quad \text{on } \partial \Omega.\]

We have obtained the homogenized problem
\[-\nabla \cdot (b \nabla u (x)) = f (x) \quad \text{in } \Omega, \tag{89}\]
\[u (x) = 0 \quad \text{on } \partial \Omega,\]
where, by (86),
\[b \nabla u (x) = \int_Y a (y) (\nabla u (x) + \nabla_y u_1 (x, y)) \, dy\]
and the local problem
\[-\nabla_y \cdot (a (y) (\nabla u (x) + \nabla_y u_1 (x, y))) = 0 \quad \text{in } \Omega \times Y, \tag{90}\]
where $u$ is the weak $H^1_0(\Omega)$-limit of $\{u^\varepsilon\}$ and $u_1 \in L^2(\Omega; H^1_y(Y)/\mathbb{R})$ in two different ways.

In the two-scale convergence method we used the result from Theorem 47, that for a sequence $\{u^\varepsilon\}$ bounded in $H^1(\Omega)$ it holds, up to a subsequence, that
\[
\nabla u^\varepsilon (x) \rightharpoonup \nabla u (x) + \nabla_y u_1 (x, y), \tag{91}\]
where $u \in H^1(\Omega)$ and $u_1 \in L^2(\Omega; H^1_y(Y)/\mathbb{R})$. As we saw in the previous section, this allows us to pass to the limit in the weak form of (88) for certain choices of test functions and thereby obtain the homogenized problem (89) and the local problem (90).

In the earlier method of asymptotic expansion one assumes (see Section 4.1) that the solution $u^\varepsilon$ can be expanded in a power series in $\varepsilon$ of the form
\[
u^\varepsilon (x) = u (x) + \varepsilon u_1 \left( x, \frac{x}{\varepsilon} \right) + \varepsilon^2 u_2 \left( x, \frac{x}{\varepsilon} \right) + ..., \tag{92}\]
where $u_k$ is $Y$-periodic in the second argument. Inserting (92) into the model problem (88) and equating equal powers of $\varepsilon$ gives both the homogenized problem and the local problem, solved by $u$ and $u_1$, respectively. Actually, it
is in this process that \( u \) turns out to only depend on \( x \). Furthermore, \( u \) and \( u_1 \) are identical to the functions with the same names that appear in (91).

Let us now proceed by studying the separate terms in (92). Concerning the first term in the expansion, we know that it is the limit of the sequence \( \{u^\varepsilon\} \) of solutions to (88) in the sense that

\[
    u^\varepsilon(x) \rightharpoonup u(x) \quad \text{in} \ H^1_0(\Omega),
\]

but it remains to be explored whether \( u_1 \) also has significance in the sense of some suitable kind of limit. The truncated form

\[
    u^\varepsilon(x) \approx u(x) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right)
\]

of (92) can be written

\[
    \frac{u^\varepsilon(x) - u(x)}{\varepsilon} \approx u_1\left(x, \frac{x}{\varepsilon}\right), \tag{93}
\]

which indicates that \( \{u^\varepsilon - u\}/\varepsilon \) approaches \( u_1 \) in a sense similar to two-scale convergence. However, we cannot expect ordinary two-scale convergence, since \( \{u^\varepsilon - u\}/\varepsilon \) is not bounded in \( L^2(\Omega) \).

According to Proposition 3.2 in [AlBr], we can identify a class of continuous functions \( v : \Omega \times Y \to \mathbb{R} \) such that

\[
    F^\varepsilon(\cdot) = \frac{1}{\varepsilon} \int_\Omega v\left(x, \frac{x}{\varepsilon}\right) (\cdot) \, dx
\]

is bounded in \( H^{-1}(\Omega) \). The key properties of these functions are that they are \( Y \)-periodic for any fixed \( x \in \Omega \) and that they have integral mean value zero over \( Y \) in their second argument. Hence, for any bounded sequence \( \{\alpha^\varepsilon\} \) in \( H^1_0(\Omega) \) and

\[
    \beta^\varepsilon = \frac{1}{\varepsilon} \int_\Omega v\left(x, \frac{x}{\varepsilon}\right) \alpha^\varepsilon(x) \, dx,
\]

\( \{\beta^\varepsilon\} \) converges up to a subsequence. This means that, still up to a subsequence, the limit

\[
    \lim_{\varepsilon \to 0} \int_\Omega \frac{u^\varepsilon(x) - u(x)}{\varepsilon} v\left(x, \frac{x}{\varepsilon}\right) \, dx
\]

exists for suitable test functions \( v \).
It turns out that the choice of test functions is crucial. We will investigate this using the example from Section 1.1, i.e. (88) with $\Omega = (0, 1)$, $f(x) = x^2$ and

$$a(y) = \frac{1}{2 + \sin 2\pi y}.$$  

Figure 20 shows the left-hand side together with the right-hand side of (93), where $u^\varepsilon$ and $u$ are the solutions to (88) and (89), respectively.

They apparently have similar patterns of oscillations but differ in the global tendency. The difference between these two functions, i.e.

$$g^\varepsilon(x) = \frac{u^\varepsilon(x) - u(x)}{\varepsilon} - u_1\left(x, \frac{x}{\varepsilon}\right),$$

is shown in Figure 21.
For the test function

\[ v(x, y) = x^2 \cos(2\pi y) \]

the graph of \( g^\varepsilon(x) v(x, \frac{x}{\varepsilon}) \) is shown in Figure 22.

Figure 22. The product \( g^\varepsilon(x) v(x, \frac{x}{\varepsilon}) \), \( \int_Y v(x, y) \, dy = 0 \).

Obviously, the integral of \( g^\varepsilon(x) v(x, \frac{x}{\varepsilon}) \) over \( \Omega \) is close to zero for small values of \( \varepsilon \). The high-frequency variations of \( g^\varepsilon \) are of vanishing amplitude; hence, the effect of these on the integral tends to zero. Furthermore, the function \( v \) has mean value zero in its second variable; thus, for small \( \varepsilon \) the slower global tendency of \( g^\varepsilon \) is filtered away by the rapid oscillations in \( v(x, \frac{x}{\varepsilon}) \).

For this choice of \( v \) and for \( \varepsilon = 0.05 \) we obtain

\[
\int_\Omega \frac{w^\varepsilon(x) - u(x)}{\varepsilon} v(x, \frac{x}{\varepsilon}) \, dx \approx 0.00224 \approx \int_\Omega \int_Y u_1(x, y) v(x, y) \, dy dx \approx 0.00221,
\]

and hence a limit of two-scale type consistent with the approach of asymptotic expansion seems to be at hand for this kind of test functions.

If we omit the requirement that \( v \) should have integral mean value zero over \( Y \) and choose, e.g.

\[ v(x, y) = x^2 (3 + \cos(2\pi y)) \, , \]

we obtain the function illustrated in Figure 23, whose integral over \( \Omega \) obviously does not vanish.
In this case,
\[
\int_{\Omega} \frac{u^\varepsilon(x) - u(x)}{\varepsilon} v \left( x, \frac{x}{\varepsilon} \right) \, dx \approx -0.02441 \\
\int_{\Omega} \int_{Y} u_1(x,y)v(x,y) \, dydx \approx 0.00221
\]
for \( \varepsilon = 0.05 \). This shows that the class of test functions must be more restricted than for usual two-scale convergence. Our investigation above reveals that \( \{ \frac{u^\varepsilon-u}{\varepsilon} \} \) is approaching \( u_1 \) only in a certain weak sense, namely
\[
\lim_{\varepsilon \to 0} \int_{\Omega} \frac{u^\varepsilon(x) - u(x)}{\varepsilon} v \left( x, \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_{Y} u_1(x,y)v(x,y) \, dydx,
\]
where the delicate question is to identify the appropriate class of test functions.

In [HSW], it is proven that for a bounded sequence \( \{u^\varepsilon\} \) in \( H^1(\Omega) \) and with \( u \) and \( u_1 \) defined as in Theorem 47, it holds that, up to a subsequence,
\[
\lim_{\varepsilon \to 0} \int_{\Omega} \frac{u^\varepsilon(x) - u(x)}{\varepsilon} v_1 \left( x, \frac{x}{\varepsilon} \right) v_2 \left( \frac{x}{\varepsilon} \right) \, dx = \int_{\Omega} \int_{Y} u_1(x,y)v_1(x)v_2(y) \, dydx \quad (94)
\]
for all \( v_1 \in D(\Omega) \) and \( v_2 \in C_0^\infty(Y)/\mathbb{R} \). Thus, \( u_1 \) appears as a limit of two-scale type to \( \{ \frac{u^\varepsilon-u}{\varepsilon} \} \), which contributes to the interpretation of the asymptotic expansion (92).

More general versions of this result (see Theorems 63 and 68) will be used to prove the homogenization results in Sections 5.3.2 and 5.3.3. For problems with more than one micro-scale, higher order terms in a multiscale expansion can be identified as limits in a way similar to (94); see [HSW].
Finally, we note that the limit $u_1$ is reminiscent of an ordinary two-scale limit in the sense that it captures the local oscillations in $\frac{u_\varepsilon - u}{\varepsilon}$; see Figure 20, but, as observed earlier, the global tendency in $x$ is not caught.

Remark 48 In the investigations in this section we have used test functions that are not zero on the boundary. Thus, they do not belong to $D(\Omega)$ as in the convergence result above. However, if one assumes that $u$ and $u_\varepsilon$ have the same boundary conditions, the factor $\frac{u_\varepsilon - u}{\varepsilon}$ will be zero on the boundary, and under these conditions the convergence result will also hold true. This is the case, for example, if $u_\varepsilon$ belongs to $H^1_0(\Omega)$.

Remark 49 The considerations in this section are made in the context of the homogenization of the model problem, where \{${u_\varepsilon}$\} converges weakly in $H^1_0(\Omega)$, and hence strongly in $L^2(\Omega)$. A relation between two-scale convergence and asymptotic expansions, which is not revealed in this connection, is that any function that admits a two-scale asymptotic expansion of the type (40)—with $u_k(x,y)$ smooth and also $Y$-periodic in $y$—two-scale converges to the first term in the expansion, i.e. to $u_0(x,y)$; see [All1].

5.2 Homogenization with multiple scales

We have already encountered homogenization problems with several microscopic scales in Section 4.3, where we considered such problems in the context of classical homogenization methods and $G$-convergence techniques. As we saw in the discussion in Section 5.1.2, the method of two-scale convergence is an efficient tool for homogenizing PDEs with periodic oscillations on one micro-scale. In this section, we will see how this method can be extended to problems with several micro-scales.
Adjusting our model problem to the case of multiple scales, we arrive at

\[-\nabla \cdot \left( a \left( \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n} \right) \nabla u^\varepsilon (x) \right) = f (x) \quad \text{in } \Omega, \]
\[u^\varepsilon (x) = 0 \quad \text{on } \partial \Omega, \quad (95)\]

where \( f \in L^2(\Omega) \), and where \( a (y_1, \ldots, y_n) \) belongs to \( \mathcal{M}(\alpha, \beta, \mathbb{R}^{nN}) \) and is \( Y_k \)-periodic in the argument \( y_k, \ k = 1, 2, ..., n \).

Here, all \( \varepsilon_k \) are supposed to be functions depending on the common variable \( \varepsilon \). The way in which the scales are related to each other is of significance, for example, for the proofs of the compactness results for gradients in Theorem 51 (Section 5.2.1) and Theorem 62 (Section 5.3.1). Apart from the assumption that the scales are microscopic, i.e. that

\[\lim_{\varepsilon \to 0} \varepsilon_k = 0\]

for \( k = 1, 2, ..., n \), we assume from now on that the scales \( \varepsilon_k \) are separated, i.e. that

\[\lim_{\varepsilon \to 0} \frac{\varepsilon_{k+1}}{\varepsilon_k} = 0\]

for \( k = 1, 2, ..., n - 1 \). This means that even though they are all microscopic, they can be distinguished from one another, making it meaningful to refer to different scales. We will also use the notion of well-separated scales, which means that there exists an integer \( m > 0 \) such that

\[\lim_{\varepsilon \to 0} \frac{1}{\varepsilon_k} \left( \frac{\varepsilon_{k+1}}{\varepsilon_k} \right)^m = 0\]

for \( k = 1, 2, ..., n - 1 \).

One can show that, under the assumption that \( a \in \mathcal{M}(\alpha, \beta, \mathbb{R}^{nN}) \), the sequence \( \{u^\varepsilon\} \) of solutions to (95) satisfies suitable a priori estimates, i.e. boundedness in \( H^1_0(\Omega) \) and hence in \( H^1(\Omega) \), and also that for a sequence of equations (95), we have \( G \)-convergence up to a subsequence. However, it is not obvious how the homogenization procedure should be performed or what the local problems will look like. After introducing the appropriate generalization of two-scale convergence in Section 5.2.1, we perform the homogenization of a problem of this form in Section 5.2.2.
5.2.1 Multiscale convergence

For reiterated homogenization problems, when we are dealing with multiple scales of periodic oscillations, we need a generalization of two-scale convergence. The so-called multiscale convergence was first introduced by Allaire and Briane in [AlBr] for an arbitrary number of microscopic scales.

**Definition 50** A sequence \( \{u^\varepsilon\} \) in \( L^2(\Omega) \) is said to \((n+1)\)-scale converge to \( u_0 \in L^2(\Omega \times Y^n) \) if

\[
\lim_{\varepsilon \to 0} \int_{\Omega} u^\varepsilon(x) v\left(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n}\right) \, dx = \int_{\Omega} \int_{Y^n} u_0(x, y_1, ..., y_n) v(x, y_1, ..., y_n) \, dy_n ... dy_1 \, dx
\]

for all \( v \in L^2(\Omega; C^1_\#(Y^n)) \). We write

\[
u^\varepsilon(x) \xrightarrow{\text{n+1}} u_0(x, y_1, ..., y_n).
\]

A compactness result similar to that in the two-scale case holds true, namely that a bounded sequence \( \{u^\varepsilon\} \) in \( L^2(\Omega) \) possesses a subsequence that \((n+1)\)-scale converges to some limit \( u_0 \in L^2(\Omega \times Y^n) \); see Theorem 2.4 in [AlBr].

We also have the corresponding result on the multiscale convergence of sequences of gradients, proven in [AlBr].

**Theorem 51** Let \( \{u^\varepsilon\} \) be a bounded sequence in \( H^1(\Omega) \). Then there exist \( n \) functions \( u_k \in L^2(\Omega \times Y^{k-1}; H^1_\#(Y_k)) \) such that, up to a subsequence,

\[
u^\varepsilon(x) \xrightarrow{\text{n+1}} u(x)
\]

and

\[
\nabla u^\varepsilon(x) \xrightarrow{\text{n+1}} \nabla u(x) + \sum_{k=1}^{n} \nabla_{y_k} u_k(x, y_1, ..., y_k),
\]

where \( u \) is the weak \( H^1(\Omega) \)-limit of \( \{u^\varepsilon\} \).

**Proof.** See the proof of Theorem 2.6 in [AlBr].
Remark 52 If \( \{u^\varepsilon\} \) is a bounded sequence in \( L^2(\Omega) \) and (96) holds for every \( v \in D(\Omega; C_0^\infty(Y^n)) \), then \( \{u^\varepsilon\} \) \((n+1)\)-scale converges. The proof is in line with the proof of Proposition 1 in [LNW].

Remark 53 The proof of Theorem 51 is done in different ways for the cases of separated and well-separated scales, respectively, but the result is the same; see the proofs of Theorem 2.6 in [AlBr], for these two cases.

5.2.2 Reiterated homogenization using multiscale convergence

Multiscale convergence allows us to pass to the limit in the homogenization procedure of problem (95). By analogy with the two-scale case, this is done using different classes of test functions, each class being responsible for a certain scale. In this case, we will end up with a system of local problems, the solutions of which are used to characterize the coefficient in the homogenized problem.

To illustrate this method, there follows a short review of the homogenization of a problem of the type (95) following [AlBr]. A case in point, yet simple, is when we have two microscopic spatial scales with periodicities \( \varepsilon \) and \( \varepsilon^2 \), respectively, which are obviously separated, i.e. the problem

\[
-\nabla \cdot \left( a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) \nabla u^\varepsilon(x) \right) = f(x) \quad \text{in } \Omega, \\
u^\varepsilon(x) = 0 \quad \text{on } \partial\Omega.
\]

In this case the coefficients can, for two different values of \( \varepsilon \), have a pattern of oscillations such as in Figure 25.

![Figure 25. Pattern of oscillations of \( a(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}) \).](image)
For this problem, we will need two different local problems to identify the limit coefficient. We start out by giving the weak form of the problem, namely to find \( u^\varepsilon \in H^1_0(\Omega) \) such that

\[
\int_{\Omega} a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) \nabla u^\varepsilon(x) \cdot \nabla v(x) \ dx = \int_{\Omega} f(x) v(x) \ dx \tag{97}
\]

holds for all \( v \in H^1_0(\Omega) \). The sequence \( \{u^\varepsilon\} \) of solutions satisfies the appropriate a priori estimate, and thus we know that, at least for a subsequence, we have weak convergence in \( H^1_0(\Omega) \) of the solutions \( u^\varepsilon \) to some limit \( u \).

To find the homogenized problem solved by \( u \), we first choose test functions with no microscopic oscillations. By Theorem 51, we can pass to the limit in (97). Hence, returning to the classical form we achieve

\[-\nabla \cdot (b \nabla u(x)) = f(x) \quad \text{in} \ \Omega,
\]

\[u(x) = 0 \quad \text{on} \ \partial \Omega,
\]

(see Remark 54) where

\[b \nabla u(x) = \int_{Y^2} a(y_1, y_2) (\nabla u(x) + \nabla y_1 u_1(x, y_1) + \nabla y_2 u_2(x, y_1, y_2)) \ dy_2 dy_1.
\]

Next, we choose test functions that have oscillations in time with the micro-scales, more precisely

\[v(x) = \varepsilon^2 v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right),
\]

where \( v_1 \in D(\Omega) \), \( v_2 \in C_\infty(Y_1) \) and \( v_3 \in C_\infty(Y_2) / \mathbb{R} \). Thus, we have

\[
\int_{\Omega} a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) \nabla u^\varepsilon(x) \cdot \nabla \left( \varepsilon^2 v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) \right) \ dx = \int_{\Omega} f(x) \varepsilon^2 v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) \ dx.
\]

We can now proceed as we did in Section 5.1.2 for the case with one micro-scale. Carrying out the differentiation of \( v \), applying Theorem 51 and using the variational lemma, we obtain as our limit the weak form of the first local problem

\[-\nabla y_2 \cdot (a(y_1, y_2) (\nabla u(x) + \nabla y_1 u_1(x, y_1) + \nabla y_2 u_2(x, y_1, y_2))) = 0,
\]

78
which we recognize as an elliptic equation for \( u_2 \) in \( Y_2 \). In a similar way to that observed in Section 5.1.2—that the solution to the local problem represented the influence of the oscillations on the \( G \)-limit \( b \)—this means that we have characterized the contribution to \( b \) from the oscillations in the \( \varepsilon^2 \)-scale.

To capture the oscillations in the \( \varepsilon \)-scale, let us now choose test functions where we leave out oscillations corresponding to the \( \varepsilon^2 \)-scale. We choose

\[
v(x) = \varepsilon v_1(x) v_2\left(\frac{x}{\varepsilon}\right),
\]

where \( v_1 \in D(\Omega) \) and \( v_2 \in C^\infty_0(Y_1)/\mathbb{R} \), which means that we have

\[
\int_{\Omega} a\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \nabla u^\varepsilon(x) \cdot \nabla \left(\varepsilon v_1(x) v_2\left(\frac{x}{\varepsilon}\right)\right) \, dx = \int_{\Omega} f(x) \varepsilon v_1(x) v_2\left(\frac{x}{\varepsilon}\right) \, dx.
\]

Again, using Theorem 51, we obtain

\[
-\nabla y_1 \cdot \int_{Y_2} a(y_1, y_2) (\nabla u(x) + \nabla y_1 u_1(x, y_1) + \nabla y_2 u_2(x, y_1, y_2)) \, dy_2 = 0.
\]

This is the other local problem, which we observe is also an elliptic equation, but for the function \( u_1 \) in \( Y_1 \).

Hence, up to a subsequence we know that the weak \( H^1_0(\Omega) \)-limit \( u \) of \( \{u^\varepsilon\} \) solves the problem

\[
-\nabla \cdot (b \nabla u(x)) = f(x) \quad \text{in } \Omega,
\]

\[
u(x) = 0 \quad \text{on } \partial \Omega,
\]

with

\[
b \nabla u(x) = \int_{Y_2} a(y_1, y_2) (\nabla u(x) + \nabla y_1 u_1(x, y_1) + \nabla y_2 u_2(x, y_1, y_2)) \, dy_2 dy_1,
\]

where \( u_1 \in L^2(\Omega; H^1_0(Y_1)/\mathbb{R}) \) and \( u_2 \in L^2(\Omega \times Y_1; H^1_0(Y_2)/\mathbb{R}) \) are the solutions to the local problems

\[
-\nabla y_2 \cdot (a(y_1, y_2) (\nabla u(x) + \nabla y_1 u_1(x, y_1) + \nabla y_2 u_2(x, y_1, y_2))) = 0,
\]

\[
-\nabla y_1 \cdot \int_{Y_2} a(y_1, y_2) (\nabla u(x) + \nabla y_1 u_1(x, y_1) + \nabla y_2 u_2(x, y_1, y_2)) \, dy_2 = 0.
\]

Arguing as we did in Section 5.1.2, we have convergence of the entire sequence, which means that the homogenization is completed. See Theorem 2.11 in [AlBr] for the corresponding result for an arbitrary finite number of scales.
Remark 54 From our observations on the $G$-convergence of $\{a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^r})\}$, it follows that the homogenized problem can be written as

\[-\nabla \cdot (b(x) \nabla u(x)) = f(x) \quad \text{in } \Omega,
\]
\[u(x) = 0 \quad \text{on } \partial \Omega.\]

This also means that we can expect that it is possible to extract $b$ from the homogenized problem as we found it. This is done by successively using the local problems, starting with the one corresponding to the smallest scale. It turns out that $b$ is constant. For details, see the inductive homogenization formula in Corollary 2.12 in [AlBr].

Remark 55 In the examples studied in this section and in Section 5.1.2, we find the local and the homogenized problems in a straightforward way, choosing the appropriate test functions. In a nonlinear case, the corresponding procedure does not provide us with the complete characterization of the limit operator. For this purpose, one uses so-called perturbed test functions; see e.g. [All1]. This is done in the proof of Theorem 72 in Section 5.3.2.

5.3 Homogenization with multiple scales in space and time

In Chapter 4, we homogenized a monotone parabolic problem with oscillations in space on two scales. In this section we will investigate some problems where oscillations appear also in the time variable, and for such problems the approach we used in Chapter 4 is not applicable. Instead, we use multiscale convergence techniques. In Section 5.3.1 we extend the multiscale convergence to include temporal oscillations, in order to fit problems of this kind. We also make the necessary preparations for the homogenization procedures. In Section 5.3.2, we consider a possibly nonlinear case where oscillations in two spatial scales and one temporal scale appear, i.e. of the type

\[\partial_t u^\varepsilon(x, t) - \nabla \cdot a\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon\right) = f(x, t) \quad \text{in } \Omega_T,
\]
\[u^\varepsilon(x, 0) = u^0(x) \quad \text{in } \Omega, \quad (98)
\]
\[u^\varepsilon(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T).\]

We perform the homogenization of this problem for $0 < r < 3$, where we study the three cases $0 < r < 2$, $r = 2$ and $2 < r < 3$, which require different
sets of local problems in order to identify the limit operator. Restricting
ourselves to the linear case, we proceed by introducing a second fast temporal
scale. A problem with rapid oscillations in one spatial scale and two temporal
scales, that is, an equation of the form

\[ \partial_t u^\epsilon (x, t) - \nabla \cdot \left( a \left( \frac{x}{\epsilon}, \frac{t}{\epsilon^r} \right) \nabla u^\epsilon (x, t) \right) = f (x, t) \quad \text{in } \Omega_T, \]

\[ u^\epsilon (x, 0) = u^0 (x) \quad \text{in } \Omega, \quad (99) \]

\[ u^\epsilon (x, t) = 0 \quad \text{on } \partial \Omega \times (0, T), \]

is studied in Section 5.3.3, where we carry out the homogenization procedure
for \( r > 0, r \neq 1 \). Also for this problem, we distinguish three different cases:
0 < \( r < 2 \), where \( r \neq 1 \), \( r = 2 \) and \( r > 2 \).

An example of the pattern of oscillations is shown in Figure 26 for the
case with oscillations in two spatial scales and one temporal scale.

![Figure 26. Oscillations in space and time.](image)

If we permute the \( x \) and \( t \) axes, this could also illustrate what the coefficient
in (99) might look like.

### 5.3.1 Evolution multiscale convergence

In the problems above, we have oscillations in space as well as in time. This
suggests that we will need a further generalization of the notion of multiscale
convergence, where there may also be temporal oscillations. We give the
following definition of evolution multiscale convergence, which includes \( n + 1 \)
spatial scales and \( m + 1 \) temporal scales, i.e. \( n \) and \( m \) micro-scales in space
and time, respectively.
**Definition 56** A sequence \( \{ u^\varepsilon \} \) in \( L^2(\Omega_T) \) is said to \((n + 1, m + 1)\)-scale converge to \( u_0 \in L^2(\Omega_T \times \mathcal{Y}_{n,m}) \) if

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} u^\varepsilon(x, t) v \left( x, t, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon_1}, \ldots, \frac{t}{\varepsilon'_m} \right) \, dx \, dt = \int_{\Omega_T} \int_{\mathcal{Y}_{n,m}} u_0(x, t, y_1, \ldots, y_n, s_1, \ldots, s_m) \cdot v(x, t, y_1, \ldots, y_n, s_1, \ldots, s_m) \, dy_n \ldots dy_1 \, ds \ldots ds_1 \, dx \, dt
\]

for all \( v \in L^2(\Omega_T; C_z(\mathcal{Y}_{n,m})) \). This is denoted by

\[
u^\varepsilon(x, t) \xrightarrow{n+1,m+1} u_0(x, t, y_1, \ldots, y_n, s_1, \ldots, s_m)\).

**3,2-scale convergence**

Let us first prepare for the homogenization of problem (98) where we have two microscopic scales in space and one fast temporal scale. Thus, adjusting for this problem we choose \( n = 2 \) and \( m = 1 \), which means that the appropriate form of evolution multiscale convergence in this case is 3,2-scale convergence. Obviously, for a sequence \( \{ u^\varepsilon \} \) in \( L^2(\Omega_T) \), 3,2-scale convergence to \( u_0 \in L^2(\Omega_T \times \mathcal{Y}_{2,1}) \) means that

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} u^\varepsilon(x, t) v \left( x, t, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2}, \frac{t}{\varepsilon_1} \right) \, dx \, dt = \int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} u_0(x, t, y_1, y_2, s) \cdot v(x, t, y_1, y_2, s) \, dy_2 \, dy_1 \, ds \, dx \, dt
\]

for all \( v \in L^2(\Omega_T; C_z(\mathcal{Y}_{2,1})) \), and the notation of this is

\[
u^\varepsilon(x, t) \xrightarrow{3,2} u_0(x, t, y_1, y_2, s)\).

The following theorem on a special case of 3,2-scale convergence, where the scales are chosen as in (98), will be useful in the sequel.

**Theorem 57** Let \( \{ u^\varepsilon \} \) be a bounded sequence in \( L^2(\Omega_T) \) and assume that we have \( \varepsilon_1 = \varepsilon, \varepsilon_2 = \varepsilon^2 \) and \( \varepsilon'_1 = \varepsilon^r \), where \( r > 0 \). Then there exists a function \( u_0 \in L^2(\Omega_T \times \mathcal{Y}_{2,1}) \) such that, for a subsequence, it holds that \( u^\varepsilon \xrightarrow{3,2} u_0 \).

**Proof.** This follows directly from Theorem 2.4 in [AlBr].
As we saw in Section 5.2.2, the result on the multiscale limit of the gradients was crucial to be able to pass to the limit in the homogenization procedure. In the next section, where we perform the homogenization of the parabolic problem (98), we need a similar result, adapted to this time-dependent case. This is stated in Theorem 62 below.

To prove this theorem, we need a few preliminaries. We will, for example, make use of a property similar to (82), i.e. that \( \{v(x, \frac{x}{\varepsilon})\} \) converges weakly in \( L^2(\Omega) \) to the mean of \( v \) in the local variable over \( Y \) if \( v \) is an admissible test function.

If we assume that \( v(x, y_1, ..., y_n) \) is \( Y_k \)-periodic in \( y_k \) for all \( k = 1, 2, ..., n \), and sufficiently smooth and the scales are separated, then according to [AlBr],

\[
v \left( x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n} \right) \rightharpoonup \int_{\mathcal{Y}_n} v(x, y_1, ..., y_n) \, dy_n...dy_1 \quad \text{in } D'(\Omega).
\]

Moreover, if \( \{v(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n})\} \) is bounded in \( L^2(\Omega) \), we obtain

\[
v \left( x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n} \right) \rightarrow \int_{\mathcal{Y}_n} v(x, y_1, ..., y_n) \, dy_n...dy_1 \quad \text{in } L^2(\Omega).
\]

A transition to our evolution setting is given in the following proposition, where the function \( v \) has oscillations in time as well as in space.

**Proposition 58** For every \( v \in C(\bar{\Omega}_T; C^\prime(\mathcal{Y}_{2,1})) \) and \( r > 0 \), it holds that

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} v \left( x, t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r} \right) \phi(x, t) \, dxdt = \int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} v(x, t, y_1, y_2, s) \phi(x, t) \, dy_2 dy_1 ds dx dt
\]

for all \( \phi \in L^2(\Omega_T) \).

**Proof.** See Proposition 5 and Remark 6 in [FIO11]. See also Corollario 3.5 in [Don] and Lemma 4.2.2 in [Pan].

To prove the result on the convergence of gradients, we also need the correspondence to (83), i.e. that a strongly convergent sequence two-scale converges to its strong limit. We have the following proposition concerning 3,2-scale convergence.
**Proposition 59** Assume that \( \{u^\varepsilon\} \) converges strongly to \( u \) in \( L^2(\Omega_T) \) and that \( \varepsilon_1 = \varepsilon, \varepsilon_2 = \varepsilon^2 \) and \( \varepsilon_1' = \varepsilon^r \), where \( r > 0 \). Then \( \{u^\varepsilon\} \) 3,2-scale converges to \( u \).

**Proof.** According to Proposition 58, it holds that

\[
v\left(x, t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}\right) \to \int_{\mathcal{Y}_{2,1}} v(x, t, y_1, y_2, s) \, dy_2 dy_1 ds \quad \text{in} \ L^2(\Omega_T)
\]

for all \( v \in C(\bar{\Omega}_T; C_\varepsilon(\mathcal{Y}_{2,1})) \). By assumption,

\[
w^\varepsilon(x, t) \to u(x, t) \quad \text{in} \ L^2(\Omega_T)
\]

and thus we have

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} u^\varepsilon(x, t) v\left(x, t, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}\right) \, dx dt = \\
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} u(x, t) v(x, t, y_1, y_2, s) \, dy_2 dy_1 ds dx dt
\]

for all \( v \in C(\bar{\Omega}_T; C_\varepsilon(\mathcal{Y}_{2,1})) \), and hence (see Remark 52) for all \( v \in L^2(\Omega_T; C_\varepsilon(\mathcal{Y}_{2,1})) \).  

Finally, we will need the following two lemmas.

**Lemma 60** Let \( H \) be the space of generalized divergence-free functions in \( L^2(\Omega; L^2_Y(Y)) \) defined by

\[
H = \left\{ v \in L^2(\Omega; L^2_Y(Y)) \, | \, \nabla_{y_2} \cdot v = 0, \int_{Y_2} \nabla_{y_1} \cdot v(x, y_1, y_2) \, dy_2 = 0 \right\}.
\]

\( H \) has the following properties:

(i) \( D(\Omega; C^\infty_Y(Y)) \cap H \) is dense in \( H \).

(ii) The orthogonal of \( H \) is

\[
H^\perp = \{ \nabla_{y_1} u_1 + \nabla_{y_2} u_2 \mid u_1 \in L^2(\Omega; H^1_\varepsilon(Y_1)), u_2 \in L^2(\Omega \times Y_1; H^1_\varepsilon(Y_2)) \}.
\]

**Proof.** See Lemma 3.7 in [AlBr].
Lemma 61 Let \( \phi \in D(\Omega; C^\infty_\#(Y^2)) \) be a function such that
\[
\int_{Y^2} \phi(x,y_1,y_2) \, dy_2 = 0
\]
and assume that the scales \( \varepsilon_1, \varepsilon_2 \) are well-separated. Then
\[
\frac{1}{\varepsilon_2} \phi \left( x, \frac{x}{\varepsilon_1}, \frac{x}{\varepsilon_2} \right) \text{ is bounded in } (H^1(\Omega))'.
\]

Proof. See Theorem 3.3 in [AlBr]. 

We are now prepared to give the following theorem on 3,2-scale convergence of gradients, which will be used while proving the homogenization results for equation (98), i.e. Theorems 72, 73 and 74 in Section 5.3.2.

Theorem 62 Let \( \{u^\varepsilon\} \) be a sequence bounded in \( H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega)) \) and assume that we have \( \varepsilon_1 = \varepsilon, \varepsilon_2 = \varepsilon^2 \) and \( \varepsilon'_1 = \varepsilon^r, r > 0 \). Then it holds, up to a subsequence, that
\[
\begin{align*}
  u^\varepsilon(x,t) &\to u(x,t) \quad \text{in } L^2(\Omega_T), \\
  u^\varepsilon(x,t) &\to u(x,t) \quad \text{in } L^2(0,T; H^1_0(\Omega))
\end{align*}
\]
and
\[
\nabla u^\varepsilon(x,t) \overset{3,2}{\to} \nabla u(x,t) + \nabla_y u_1(x,t,y_1,s) + \nabla_y u_2(x,t,y_1,y_2,s),
\]
where \( u \in L^2(0,T; H^1_0(\Omega)) \), \( u_1 \in L^2(\Omega_T \times (0,1); H^1_\#(Y_1)/\mathbb{R}) \) and \( u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; H^1_\#(Y_2)/\mathbb{R}) \).

Proof. Since \( \{u^\varepsilon\} \) is bounded in \( H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega)) \), it is obviously bounded in \( L^2(0,T; H^1_0(\Omega)) \). Hence, we know by Theorem 57 that \( \{u^\varepsilon\} \) and \( \{\nabla u^\varepsilon\} \) both possess a 3,2-scale limit, up to a subsequence. This means that there exist \( u_0 \in L^2(\Omega_T \times \mathcal{Y}_{2,1}) \) and \( w_0 \in L^2(\Omega_T \times \mathcal{Y}_{2,1})^N \) such that, at least for a subsequence,
\[
\begin{align*}
  u^\varepsilon(x,t) &\overset{3,2}{\to} u_0(x,t,y_1,y_2,s) \\
  \nabla u^\varepsilon(x,t) &\overset{3,2}{\to} w_0(x,t,y_1,y_2,s).
\end{align*}
\]
From the assumption of boundedness in \( H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega)) \), it also follows that \( \{\partial_t u^\varepsilon\} \) is bounded in \( L^2(0,T; H^{-1}(\Omega)) \). For \( \{u^\varepsilon\} \) bounded in
$L^2(0, T; H^1_0(\Omega))$ with $\{\partial_t u^\varepsilon\}$ bounded in $L^2(0, T; H^{-1}(\Omega))$, it is proven in Lemmas 8.2 and 8.4 in [CoFo] that for a suitable subsequence

$$u^\varepsilon(x, t) \to u(x, t) \quad \text{in } L^2(\Omega_T).$$

(100)

See also Problem 23.13b in [ZeiIIA]. Furthermore, the boundedness in $L^2(0, T; H^1_0(\Omega))$ gives that, up to a subsequence,

$$u^\varepsilon(x, t) \to u(x, t) \quad \text{in } L^2(0, T; H^{-1}_0(\Omega)).$$

By (100), we can apply Proposition 59 which yields

$$\int_{\Omega_T} u^\varepsilon(x, t) v\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) c_1(t) c_2\left(\frac{t}{\varepsilon^r}\right) \, dx \, dt \to$$

$$\int_{\Omega_T} u(x, t) \left(\int_{\mathcal{Y}_{2,1}} v(x, y_1, y_2) c_1(t) c_2(s) \, dy_2 dy_1 ds\right) \, dx \, dt =$$

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} u(x, t) v(x, y_1, y_2) c_1(t) c_2(s) \, dy_2 dy_1 ds \, dx \, dt$$

for all $c_1 \in D(0, T)$, $c_2 \in C^\infty_c(0, 1)$ and $v \in D(\Omega; C^\infty_c(Y^2))^N$.

Our next aim is to characterize the limit $w_0$. For this purpose, we choose functions $v$ belonging to the set $D(\Omega; C^\infty_c(Y^2))^N \cap H$ introduced in Lemma 60. We know that, up to a subsequence,

$$\int_{\Omega_T} \nabla u^\varepsilon(x, t) \cdot v\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) c_1(t) c_2\left(\frac{t}{\varepsilon^r}\right) \, dx \, dt \to$$

$$\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} w_0(x, t, y_1, y_2, s) \cdot v(x, y_1, y_2) c_1(t) c_2(s) \, dy_2 dy_1 ds \, dx \, dt$$

and we carry out the characterization by studying the limit process for

$$\int_{\Omega_T} \nabla u^\varepsilon(x, t) \cdot v\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) c_1(t) c_2\left(\frac{t}{\varepsilon^r}\right) \, dx \, dt.$$

For $t$ fixed, integration by parts over $\Omega$ gives

$$-\int_{\Omega} u^\varepsilon(x, t) \left(\nabla + \varepsilon^{-1}\nabla y_1 + \varepsilon^{-2}\nabla y_2\right) \cdot v\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) c_1(t) c_2\left(\frac{t}{\varepsilon^r}\right) \, dx =$$

$$-\int_{\Omega} u^\varepsilon(x, t) \left(\nabla + \varepsilon^{-1}\nabla y_1\right) \cdot v\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) c_1(t) c_2\left(\frac{t}{\varepsilon^r}\right) \, dx,$$
due to the definition of $H$. We will now see that the contribution from the term
\[
\int_{\Omega} u^\varepsilon(x, t) \varepsilon^{-1} \nabla y_1 \cdot v \left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) c_1(t) c_2 \left(\frac{t}{\varepsilon^r}\right) \, dx
\]
tends to zero. If $\phi \in D \left(\Omega; C^\infty_\varepsilon(Y^2)\right)$ and
\[
\int_{Y_2} \phi(x, y_1, y_2) \, dy_2 = 0,
\]
then $\varepsilon^{-2} \phi(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2})$ is bounded in $(H^1(\Omega))'$, by Lemma 61, i.e.
\[
\left| \int_{\Omega} \varepsilon^{-2} \phi \left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \rho(x) \, dx \right| \leq C \|\rho\|_{H^1(\Omega)}
\]
for all $\rho \in H^1(\Omega)$. Since $\nabla y_1 \cdot v$ fulfills the conditions on $\phi$ above, we get
\[
\left| \int_{\Omega} \varepsilon^{-1} \nabla y_1 \cdot v \left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) \rho(x) \, dx \right| \leq C \|\rho\|_{H^1(\Omega)},
\]
that is
\[
\left| \int_{\Omega} \varepsilon^{-1} \nabla y_1 \cdot v \left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) \rho(x) \, dx \right| \leq C \varepsilon \|\rho\|_{H^1(\Omega)}.
\]
For $t$ fixed and $\rho = u^\varepsilon$, we have
\[
\left| \int_{\Omega} \varepsilon^{-1} \nabla y_1 \cdot v \left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) u^\varepsilon(x, t) \, dx \right| \leq C \varepsilon \|u^\varepsilon\|_{H^1(\Omega)}
\]
and it follows that
\[
\left| \int_{\Omega T} \varepsilon^{-1} \nabla y_1 \cdot v \left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) u^\varepsilon(x, t) \, dx dt \right|^2 \leq
\]
\[
C_1 \int_0^T \left| \int_{\Omega} \varepsilon^{-1} \nabla y_1 \cdot v \left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) u^\varepsilon(x, t) \, dx \right|^2 dt \leq
\]
\[
C_2 \varepsilon^2 \int_0^T \|u^\varepsilon (\cdot, t)\|_{H^1(\Omega)}^2 \, dt = C_2 \varepsilon^2 \|u^\varepsilon\|_{L^2(0,T;H^1(\Omega))}^2 \leq C_3 \varepsilon^2 \rightarrow 0
\]
as $\varepsilon \to 0$. We have proven that
\[
- \int_{\Omega_T} u^\varepsilon(x, t) \left(\nabla + \varepsilon^{-1} \nabla y_1 + \varepsilon^{-2} \nabla y_2\right) \cdot v \left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2} \right) c_1(t) c_2 \left(\frac{t}{\varepsilon^r}\right) \, dx dt \to
\]
\[
- \int_{\Omega_T} \int_{Y_{2,1}} u(x, t) \nabla \cdot v(x, y_1, y_2) c_1(t) c_2(s) \, dy_2 dy_1 ds \, dx dt =
\]

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\[
\int_{\Omega_T} \int_{Y_2,1} \nabla u(x,t) \cdot v(x,y_1,y_2) c_1(t) c_2(s) \ dy_2 dy_1 ds dx dt
\]
for all \( v \in D(\Omega; C^\infty_4(Y^2))^N \cap H \), all \( c_1 \in D(0,T) \) and all \( c_2 \in C^\infty_4(0,1) \). Thus, we have

\[
\int_{\Omega_T} \int_{Y_2,1} w_0(x,t,y_1,y_2,s) \cdot v(x,y_1,y_2) c_1(t) c_2(s) \ dy_2 dy_1 ds dx dt =
\int_{\Omega_T} \int_{Y_2,1} \nabla u(x,t) \cdot v(x,y_1,y_2) c_1(t) c_2(s) \ dy_2 dy_1 ds dx dt
\]
for every such \( v, c_1 \) and \( c_2 \). By the variational lemma, we get for almost all \( s \) and \( t \)

\[
\int_{\Omega} \int_{Y^2} w_0(x,t,y_1,y_2,s) \cdot v(x,y_1,y_2) \ dy_2 dy_1 dx =
\int_{\Omega} \int_{Y^2} \nabla u(x,t) \cdot v(x,y_1,y_2) \ dy_2 dy_1 dx
\]
for all \( v \in D(\Omega; C^\infty_4(Y^2))^N \cap H \) and by density (see (i) in Lemma 60), this holds for all \( v \in H \). Hence, \( w_0 \) and \( \nabla u \) can only differ by elements in \( H^\perp \), that is, we have

\[
w_0(x,t,y_1,y_2,s) - \nabla u(x,t) = \nabla_{y_1} u_1(x,t,y_1,s) + \nabla_{y_2} u_2(x,t,y_1,y_2,s)
\]
for almost all \( s \) and \( t \); that is

\[
w_0(x,t,y_1,y_2,s) = \nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s) + \nabla_{y_2} u_2(x,t,y_1,y_2,s).
\]
It remains for us to localize \( u_1 \) and \( u_2 \) into suitable function spaces. If we choose \( u_1 \) and \( u_2 \) with average values of zero over \( Y_1 \) and \( Y_2 \), respectively, this means that we should prove that \( u_1 \in L^2(\Omega_T \times (0,1); H^1_2(Y_1)/\mathbb{R}) \) and \( u_2 \in L^2(\Omega_T \times Y_1; H^1_2(Y_2)/\mathbb{R}) \). We have, for all \( v \in D(\Omega_T, C^\infty_4(Y_{1,1}))^N \),

\[
\int_{\Omega_T} \int_{Y_{2,1}} (w_0(x,t,y_1,y_2,s) - \nabla u(x,t)) \cdot v(x,t,y_1,s) \ dy_2 dy_1 ds dx dt =
\int_{\Omega_T} \int_{Y_{2,1}} (\nabla_{y_1} u_1(x,t,y_1,s) + \nabla_{y_2} u_2(x,t,y_1,y_2,s)) \cdot v(x,t,y_1,s) \ dy_2 dy_1 ds dx dt =
\int_{\Omega_T} \int_{Y_{1,1}} \nabla_{y_1} u_1(x,t,y_1,s) \cdot v(x,t,y_1,s) \ dy_1 ds dx dt,
\]

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Proof. See Corollary 3.3 in [Hol2] and Theorem 3 in [HSW], respectively, by the Poincaré-Wirtinger inequality.

\[ \nabla_{y_1} u_1(x, t, y_1, s) = \int_{Y_2} w_0(x, t, y_1, y_2, s) \, dy_2 - \nabla u(x, t) \]

and hence \( \nabla_{y_1} u_1 \in L^2(\Omega_T \times \mathcal{Y}_{1,1})^N \), that is, \( u_1 \in L^2(\Omega_T \times (0, 1); H^1_z(Y_1)/\mathbb{R}) \). It follows that

\[ \nabla_{y_2} u_2(x, t, y_1, y_2, s) = w_0(x, t, y_1, y_2, s) - \nabla u(x, t) - \nabla_{y_1} u_1(x, t, y_1, s), \]

i.e. \( \nabla_{y_2} u_2 \in L^2(\Omega_T \times \mathcal{Y}_{2,1})^N \), and hence \( u_2 \in L^2(\Omega_T \times \mathcal{Y}_{1,1}; H^1_z(Y_2)/\mathbb{R}) \). The functions \( u_1 \) and \( u_2 \) are bounded in \( L^2(\Omega_T \times \mathcal{Y}_{1,1}) \) and \( L^2(\Omega_T \times \mathcal{Y}_{2,1}) \), respectively, by the Poincaré-Wirtinger inequality. □

For the homogenization procedure in the next section we also need the following theorem, which generalizes our observations in Section 5.1.3 to a particular evolution case. A similar result, also including non-periodic cases, has been proven recently by Nguetseng and Woukeng in [NgWo2].

**Theorem 63** Let \( \{u^\varepsilon\} \) be a bounded sequence in \( H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega)) \), and let \( u \) and \( u_1 \) be defined as in Theorem 62. Then, up to a subsequence,

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \frac{u^\varepsilon(x, t) - u(x, t)}{\varepsilon} v_1(x) v_2\left(\frac{x}{\varepsilon}\right) c_1(t) c_2\left(\frac{t}{\varepsilon^r}\right) \, dxdt = \\
\int_{\Omega_T} \int_{\mathcal{Y}_{1,1}} u_1(x, t, y_1, s) v_1(x) v_2(y_1) c_1(t) c_2(s) \, dy_1 ds dxdt
\]

for all \( v_1 \in D(\Omega) \), \( v_2 \in L^2_z(Y_1)/\mathbb{R} \), \( c_1 \in D(0, T) \) and \( c_2 \in L^2_z(0, 1) \).

**Proof.** See Corollary 3.3 in [Hol2] and Theorem 3 in [HSW]. □

Our last preparation concerning problem (98) is the following proposition, proven in [NgWo2]. This will be used for the case where \( r = 2 \).

**Proposition 64** Let \( u \) and \( v \) belong to \( L^2_z(0, 1; H^1_z(Y)/\mathbb{R}) \) and assume that \( \partial_s u \) and \( \partial_s v \) are in \( L^2_z(0, 1; (H^1_z(Y)/\mathbb{R})') \). Then

\[
\int_0^1 \langle \partial_s u(s), v(s) \rangle_{(H^1_z(Y)/\mathbb{R})', (H^1_z(Y)/\mathbb{R})} \, ds + \int_0^1 \langle \partial_s v(s), u(s) \rangle_{(H^1_z(Y)/\mathbb{R})', (H^1_z(Y)/\mathbb{R})} \, ds = 0.
\]

**Proof.** See Corollary 4.1. in [NgWo2]. □
The results we have established so far in this section have been with a view to homogenizing (98). We also need similar results to prepare for the homogenization of (99). Adapting to this problem, we obtain 2,3-scale convergence that includes two spatial scales and three temporal scales. Following the general definition of evolution multiscale convergence, 2,3-scale convergence of a sequence \( \{u^\varepsilon\} \) in \( L^2(\Omega_T) \) to a function \( u_0 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}) \) means that

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} u^\varepsilon(x,t) v \left( x, t, \frac{x}{\varepsilon_1}, \frac{t}{\varepsilon_2}, \frac{t}{\varepsilon_3} \right) \, dx \, dt = \int_{\Omega_T} \int_{\mathcal{Y}_{1,2}} u_0(x,t,y,s_1,s_2) v(x,t,y,s_1,s_2) \, dy \, ds_2 ds_1 \, dx \, dt
\]

for all \( v \in L^2(\Omega_T; C^1(\mathcal{Y}_{1,2})) \). We denote this by

\[
u^\varepsilon(x,t) \overset{2,3}{\to} u_0(x,t,y,s_1,s_2).
\]

The following compactness result holds true.

**Theorem 65** Let \( \{u^\varepsilon\} \) be a bounded sequence in \( L^2(\Omega_T) \) and assume that \( \varepsilon_1 = \varepsilon, \varepsilon'_1 = \varepsilon \) and \( \varepsilon'_2 = \varepsilon^r \), where \( r > 0 \) and \( r \neq 1 \). Then there exists a function \( u_0 \in L^2(\Omega_T \times \mathcal{Y}_{1,2}) \) such that, up to a subsequence, \( \{u^\varepsilon\} \) 2,3-scale converges to \( u_0 \).

**Proof.** This is an immediate consequence of Theorem 2.4 in [AlBr].

**Remark 66** We omit the case \( r = 1 \) to obtain separation between the scales, see [AlBr] for details.

We also have the theorem below on 2,3-scale convergence of gradients.

**Theorem 67** Let \( \{u^\varepsilon\} \) be a sequence bounded in \( H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega)) \) and assume that \( \varepsilon_1 = \varepsilon, \varepsilon'_1 = \varepsilon \) and \( \varepsilon'_2 = \varepsilon^r \), where \( r > 0 \) and \( r \neq 1 \). Then it holds, up to a subsequence, that

\[
u^\varepsilon(x,t) \overset{2,3}{\to} u(x,t) \quad \text{in} \quad L^2(\Omega_T),
\]

\[
u^\varepsilon(x,t) \overset{2,3}{\to} u(x,t) \quad \text{in} \quad L^2(0,T; H^1_0(\Omega))
\]

and

\[
\nabla u^\varepsilon(x,t) \overset{2,3}{\to} \nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2),
\]

where \( u \in L^2(0,T; H^1_0(\Omega)) \) and \( u_1 \in L^2(\Omega_T \times (0,1)^2; H^1_2(Y)/\mathbb{R}) \).
The proof follows directly along the lines of the proof of Theorem 3.1 in [Hol2].

Finally, we have the following correspondence to Theorem 63.

**Theorem 68** Let \( \{u^\varepsilon\} \) be a bounded sequence in \( H^1(0,T; H^1_0(\Omega), H^{-1}(\Omega)) \), and let \( u \) and \( u_1 \) be defined as in Theorem 67. Then, up to a subsequence,

\[
\lim_{\varepsilon \to 0} \int_{\Omega_T} \frac{u^\varepsilon(x,t) - u(x,t)}{\varepsilon} v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) c_3 \left( \frac{t}{\varepsilon^r} \right) \, dxdt = \\
\int_{\Omega_T} \int_{\mathcal{Y}_{1,2}} u_1(x,t,y,s_1,s_2) v_1(x) v_2(y) c_1(t) c_2(s_1) c_3(s_2) \, dyds_2ds_1dxdt
\]

for all \( v_1 \in D(\Omega), v_2 \in L^2_0(Y)/\mathbb{R}, c_1 \in D(0,T) \) and \( c_2, c_3 \in L^2_0(0,1) \) if \( r > 0, r \neq 1 \).

**Proof.** The proof is performed in exactly the same way as the proof of Corollary 3.3 in [Hol2].

Theorems 65, 67 and 68 above will be used while proving the homogenization result in Theorem 76 in Section 5.3.3.

**Remark 69** The first results on two-scale convergence were presented by Nguetseng in [Ngu1], where the efficiency of this homogenization technique is illustrated on linear elliptic problems. In [All1] Allaire gives a different proof for two-scale compactness, where some new sets of test functions are introduced. The method is applied to several non-trivial problems such as the homogenization of linear elliptic problems in perforated domains (see also e.g. [AlBr], [Ngu3] and [FHOSi2]) and homogenization of nonlinear elliptic problems under certain monotonicity assumptions. In [AlBr], Allaire and Briane introduce a generalization to the multiple scale case and use this for homogenization of linear elliptic problems with multiple scales; see Section 5.2. The nonlinear monotone case is studied by Lions et al. in [LLPW]. In [CaGa], the extension to the almost periodic case is treated by Casado-Diaz and Gayte, and in [BMW] Bourgeat et al. develop a type of stochastic two-scale convergence. A careful investigation of the properties of periodic two-scale convergence is made by Lakkassen et al. in [LNW]. A generalization to certain non-periodic cases is made in [Ngu2] under the name of \( \Sigma \)-convergence and a corresponding extension of the results in Theorems 63 and 68 can be found in [NgWo2]. In [CDG], the closely related approach of so-called unfolding is introduced by Cioranescu et al. This is also studied by Nechvatal in [Nec].
5.3.2 Homogenization of some monotone parabolic operators with three spatial and two temporal scales

We are now prepared to further investigate the parabolic equation

$$\partial_t u^\varepsilon(x,t) - \nabla \cdot a\left(\frac{x}{\varepsilon^1}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^3}, \nabla u^\varepsilon\right) = f(x,t) \quad \text{in } \Omega_T,$$

$$u^\varepsilon(x,0) = u^0(x) \quad \text{in } \Omega, \quad (101)$$

$$u^\varepsilon(x,t) = 0 \quad \text{on } \partial \Omega \times (0,T)$$

where $u^0 \in L^2(\Omega)$ and $f \in L^2(\Omega_T)$. The function

$$a : \mathbb{R}^{2N} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$$

is assumed to satisfy the following conditions, where $0 < k \leq 1$ and $\alpha$ and $\beta$ are positive constants:

(i) $a(y_1, y_2, s, 0) = 0$ for all $(y_1, y_2, s) \in \mathbb{R}^{2N} \times \mathbb{R}$.

(ii) $a(\cdot, \cdot, \cdot, \xi)$ is continuous and $\mathcal{Y}_{2,1}$-periodic for all $\xi \in \mathbb{R}^N$.

(iii) $a(y_1, y_2, s, \cdot)$ is continuous for all $(y_1, y_2, s) \in \mathbb{R}^{2N} \times \mathbb{R}$.

(iv) $|a(y_1, y_2, s, \xi) - a(y_1, y_2, s, \xi')| \leq \beta (1 + |\xi| + |\xi'|)^{1-k} |\xi - \xi'|^k$

for all $(y_1, y_2, s) \in \mathbb{R}^{2N} \times \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$.

(v) $a(y_1, y_2, s, \xi) - a(y_1, y_2, s, \xi') \cdot (\xi - \xi') \geq \alpha |\xi - \xi'|^2$

for all $(y_1, y_2, s) \in \mathbb{R}^{2N} \times \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$.

In a way similar to that in Section 4.3, we observe that this means that $a$ belongs to $\mathcal{N}(\alpha, \beta, k, \mathbb{R}^{2N} \times \mathbb{R})$.

**Remark 70** Note that when we studied this problem with respect to existence and uniqueness of the solution in Chapter 2 we considered $f$ in $L^2(0,T; H^{-1}(\Omega))$, whereas we now have chosen right-hand sides in $L^2(\Omega_T)$ to fit the multiscale convergence approach. It can be shown (see Example 23.4 in [ZeiIIA]) that these can be identified with functionals in $L^2(0,T; H^{-1}(\Omega))$.

We will establish the existence of a $G$-limit for the sequence of operators corresponding to (101) and also characterize this limit by performing the homogenization procedure for this problem. This will be done for different values of the constant $r$, that is for different frequencies of the time oscillations. We consider the three cases $0 < r < 2$, $r = 2$ and $2 < r < 3$.  

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Let us start by giving the following theorem.

**Theorem 71** For problem (101), it holds that

(a) the problem has a unique solution \( u^\varepsilon \in H^1 (0, T; H_0^1(\Omega), H^{-1}(\Omega)) \) for every fixed \( \varepsilon > 0 \).

(b) the solutions \( u^\varepsilon \) are uniformly bounded in \( L^\infty (0, T; L^2(\Omega)) \) and \( H^1 (0, T; H_0^1(\Omega), H^{-1}(\Omega)) \), i.e. for some positive constant \( C \)

\[
\| u^\varepsilon \|_{L^\infty(0,T;L^2(\Omega))} \leq C
\]

and

\[
\| u^\varepsilon \|_{H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))} \leq C.
\]

(c) the sequence of functions

\[
a^h(x, t, \xi) = a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \xi \right)
\]

where \((x, t, \xi) \in \Omega_T \times \mathbb{R}^N\) and \( \varepsilon = \varepsilon (h) \to 0 \) as \( h \to \infty \) belongs to \( \mathcal{N}(\alpha, \beta, k, \Omega_T) \). Hence, \( \{a^h\} G\)-converges up to a subsequence, to a limit \( b \in \mathcal{N}(\alpha, \tilde{\beta}, \tilde{k}, \Omega_T) \), where \( \tilde{k} = k/(2 - k) \) and \( \tilde{\beta} \) is a positive constant depending on \( \alpha, \beta \) and \( k \) only.

**Proof.** The proof of the result (a) on the existence of a unique solution is carried out in Section 2.2.2, Theorem 8. The proof of (b) and (c) is done in an analogous way to the proof of Theorem 41. See also [FlOl2].

The result in (a) means that we have a unique weak solution to (101), i.e. that there is a unique \( u^\varepsilon \in H^1 (0, T; H_0^1(\Omega), H^{-1}(\Omega)) \) for every \( \varepsilon > 0 \) such that \( u^\varepsilon (x, 0) = u^0(x) \) and

\[
\int_{\Omega_T} -u^\varepsilon (x, t) v (x) \partial_t c (t) + a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \cdot \nabla v (x) c (t) \, dx dt = \quad (102)
\]

\[
\int_{\Omega_T} f (x, t) v (x) c (t) \, dx dt
\]

for all \( v \in H_0^1(\Omega) \) and \( c \in D (0, T) \).
We also know from the $G$-convergence result in (c) that the sequence $\{u^\varepsilon\}$ of solutions converges weakly in $L^2(0, T; H^1_0(\Omega))$, up to a subsequence, to a limit $u$, which solves a well-posed parabolic problem. What remains to be done is to determine the $G$-limit $b$. We will do this by applying a generalization of the multiscale convergence method described in Section 5.2.2, and we will see that it can be done by means of local problems reminiscent of those found there. We first consider the case when $0 < r < 2$.

**Theorem 72** Let $\{u^\varepsilon\}$ be a sequence of solutions in $H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega))$ to (101) for $0 < r < 2$. Then it holds that

\[
u^\varepsilon(x, t) \rightarrow u(x, t) \quad \text{in } L^2(\Omega_T), \quad u^\varepsilon(x, t) \rightarrow u(x, t) \quad \text{in } L^2(0, T; H^1_0(\Omega))
\]

and

\[
\nabla u^\varepsilon(x, t) \rightarrow \nabla u(x, t) + \nabla_{y_1} u_1(x, t, y_1, s) + \nabla_{y_2} u_2(x, t, y_1, y_2, s)
\]

where $u \in H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega))$ is the unique solution to the homogenized problem

\[
\begin{align*}
\partial_t u(x, t) - \nabla \cdot b(x, t) \nabla u &= f(x, t) \quad \text{in } \Omega_T, \\
u(x, 0) &= u^0(x) \quad \text{in } \Omega, \\
u(x, t) &= 0 \quad \text{on } \partial \Omega \times (0, T),
\end{align*}
\]

where in turn

\[
b(x, t) = \int_{\mathbb{R}^2} a(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) \, dy_2 dy_1 ds.
\]

Here, $u_1 \in L^2(\Omega_T \times (0, 1); H^1_{x,1}(Y_1) / \mathbb{R})$ and $u_2 \in L^2(\Omega_T \times Y_{1,1}; H^1_y(Y_2) / \mathbb{R})$ are the unique solutions to the system of local problems

\[
\begin{align*}
-\nabla_{y_2} \cdot a(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) &= 0, \\
-\nabla_{y_1} \cdot \int_{Y_2} a(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) \, dy_2 &= 0.
\end{align*}
\]

**Proof.** By the properties of $a$, we can successively apply (b) in Theorem 71 and Theorem 62 to conclude that, up to a subsequence,

\[
u^\varepsilon(x, t) \rightarrow u(x, t) \quad \text{in } L^2(\Omega_T), \quad u^\varepsilon(x, t) \rightarrow u(x, t) \quad \text{in } L^2(0, T; H^1_0(\Omega))
\]
and
\[ \nabla u^\varepsilon(x, t) \xrightarrow{\varepsilon \to 0} \nabla u(x, t) + \nabla y_1 u_1(x, t, y_1, s) + \nabla y_2 u_2(x, t, y_1, y_2, s), \]
where \( u \in L^2(0, T; H^1_0(\Omega)), u_1 \in L^2(\Omega_T \times (0, 1); H^1_0(Y_1) / \mathbb{R}) \) and
\( u_2 \in L^2(\Omega_T \times \mathcal{Y}_1; H^1_0(Y_2) / \mathbb{R}) \). Moreover, the growth condition (29) and
the boundedness of \( \{ u^\varepsilon \} \) in \( L^2(0, T; H^1_0(\Omega)) \) imply that \( \{ a(\varepsilon, \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon) \} \) is
bounded in \( L^2(\Omega_T)^N \). Thus, by Theorem 57, there exists a function
\( a^0 \in L^2(\Omega_T \times \mathcal{Y}_{2,1})^N \) such that
\[ a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \xrightarrow{\varepsilon \to 0} a^0(x, t, y_1, y_2, s) \quad (105) \]
up to a subsequence. In (102) we choose \( v \in H^1_0(\Omega) \) and \( c \in D(0, T) \) to be
independent of \( \varepsilon \). When passing to the limit, we get
\[ \int_{\Omega_T} -u(x, t) v(x) \partial_t c(t) + \left( \int_{\mathcal{Y}_{2,1}} a^0(x, t, y_1, y_2, s) \ dy_2 dy_1 ds \right) \cdot \nabla v(x) c(t) \, dx \, dt = \quad (106) \]
\[ \int_{\Omega_T} f(x, t) v(x) c(t) \, dx \, dt, \]
which means that
\[ b(x, t, \nabla u) = \int_{\mathcal{Y}_{2,1}} a^0(x, t, y_1, y_2, s) \, dy_2 dy_1 ds. \]

The next step is to find the local problems (103) and (104), the solutions
\( u_1 \) and \( u_2 \) of which we will use to characterize \( a^0 \). We look for one of the local
problems, choosing test functions in (102) according to
\[ v(x) = \varepsilon^2 v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right), \quad c(t) = c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right), \]
where \( v_1 \in D(\Omega), v_2 \in C^\infty_\#(Y_1), v_3 \in C^\infty_\#(Y_2) / \mathbb{R}, c_1 \in D(0, T) \) and
\( c_2 \in C^\infty_2 (0, 1) \), and get
\[
\int_{\Omega_T} -u^\varepsilon (x, t) \varepsilon^2 v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) \partial_t \left( c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) + \\
\alpha \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \cdot \nabla \left( \varepsilon^2 v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dxdt = \\
\int_{\Omega_T} f (x, t) \varepsilon^2 v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dxdt.
\]

Hence,
\[
\int_{\Omega_T} -u^\varepsilon (x, t) v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) \left( \varepsilon^2 \partial_t c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) + \\
\varepsilon^2 r c_1 (t) \partial_s c_2 \left( \frac{t}{\varepsilon^r} \right) + a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \cdot \left( \varepsilon^2 \nabla v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) \right) + \\
\varepsilon v_1 (x) \nabla_y v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) + v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) \nabla_y v_3 \left( \frac{x}{\varepsilon^2} \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dxdt = \\
\int_{\Omega_T} f (x, t) \varepsilon^2 v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \, dxdt
\]

and when \( \varepsilon \) tends to zero, we obtain
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} a^0 (x, t, y_1, y_2, s) \cdot v_1 (x) v_2 (y_1) \nabla_y v_3 (y_2) c_1 (t) c_2 (s) \, dy_2 dy_1 ds \, dxdt = 0.
\]

Finally, applying the variational lemma, we get
\[
\int_{\mathcal{Y}_2} a^0 (x, t, y_1, y_2, s) \cdot \nabla_y v_3 (y_2) \, dy_2 = 0 \tag{107}
\]

almost everywhere in \( \Omega_T \times \mathcal{Y}_{1,1} \) for all \( v_3 \in C^\infty_2 (Y_2) / \mathbb{R} \), and, by density, for all \( v_3 \in H^1_2 (Y_2) / \mathbb{R} \). In what follows, the corresponding conclusion concerning density will be understood implicitly while deriving local problems. Equation (107) will turn out to be the weak form of (103).

To find the second local problem we now study the difference between (102) and (106) for test functions
\[
v (x) = \varepsilon v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right), \quad c (t) = c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right),
\]

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where \( v_1 \in D(\Omega), v_2 \in C^\infty_\sharp (Y_1) / \mathbb{R}, c_1 \in D(0, T) \) and \( c_2 \in C^\infty_\sharp (0, 1) \), and get

\[
\int_{\Omega_T} (u^\varepsilon (x, t) - u (x, t)) \varepsilon v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) \partial_t \left( c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) dx dt + \\
\int_{\Omega_T} \left( \int_{Y_{2,1}} a^0 (x, t, y_1, y_2, s) dy_2 dy_1 ds - a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \right) \cdot \nabla \left( \varepsilon v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) dx dt = \\
\int_{\Omega_T} (f (x, t) - f (x, t)) \varepsilon v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) dx dt = 0.
\]

Carrying out the differentiations and after some rewriting, we have

\[
\int_{\Omega_T} \frac{1}{\varepsilon} (u^\varepsilon (x, t) - u (x, t)) v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) \cdot \left( \varepsilon^2 \partial_t c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) + \varepsilon^{2-r} c_1 (t) \partial_s c_2 \left( \frac{t}{\varepsilon^r} \right) \right) + \\
\left( \int_{Y_{2,1}} a^0 (x, t, y_1, y_2, s) dy_2 dy_1 ds - a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \right) \left( \varepsilon \nabla v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) + v_1 (x) \nabla y_1 v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) dx dt = 0.
\]

For \( 0 < r < 2 \) we obtain by Theorem 63, or alternatively by Lemma 61, that the first term vanishes, and by (105) we get

\[
\int_{\Omega_T} \int_{Y_{2,1}} \left( \int_{Y_{2,1}} a^0 (x, t, y_1, y_2, s) dy_2 dy_1 ds - a^0 (x, t, y_1, y_2, s) \right) \cdot v_1 (x) \nabla y_1 v_2 (y_1) c_1 (t) c_2 (s) \ dy_2 dy_1 ds dx dt = 0
\]

as \( \varepsilon \) tends to zero. Due to the \( Y_1 \)-periodicity of \( v_2 \) and the variational lemma, we obtain

\[
\int_{Y_1} \left( \int_{Y_2} a^0 (x, t, y_1, y_2, s) \ dy_2 \right) \cdot \nabla y_1 v_2 (y_1) \ dy_1 = 0 \quad (108)
\]

for all \( v_2 \in H^1_\sharp (Y_1) / \mathbb{R} \), i.e. we have the weak form of (104) if we can prove that

\[
a^0 (x, t, y_1, y_2, s) = a (y_1, y_2, s, \nabla u + \nabla y_1 u_1 + \nabla y_2 u_2).
\]
For the characterization of the limit \( a^0 \), we use perturbed test functions

\[
p^k (x, t, y_1, y_2, s) = p^{k,0} (x, t) + p^{k,1} (x, t, y_1, s) + p^{k,2} (x, t, y_1, y_2, s) + \delta c (x, t, y_1, y_2, s) ,
\]

where \( p^{k,0} \in D (\Omega_T)^N, p^{k,1} \in D (\Omega_T; C^\infty (Y_{1,1}))^N, p^{k,2}, c \in D (\Omega_T; C^\infty (Y_{2,1}))^N \) and \( \delta > 0 \). These sequences are chosen such that

\[
p^{k,0} (x, t) \rightarrow \nabla u (x, t) \quad \text{in} \quad L^2 (\Omega_T)^N ,
\]

\[
p^{k,1} (x, t, y_1, s) \rightarrow \nabla_{y_1} u_1 (x, t, y_1, s) \quad \text{in} \quad L^2 (\Omega_T \times Y_{1,1})^N
\]

and

\[
p^{k,2} (x, t, y_1, y_2, s) \rightarrow \nabla_{y_2} u_2 (x, t, y_1, y_2, s) \quad \text{in} \quad L^2 (\Omega_T \times Y_{2,1})^N,
\]

and such that they converge almost everywhere to the same limits as \( k \rightarrow \infty \).

We define

\[
p^k_\varepsilon (x, t) = p^k \left( x, t, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r} \right).
\]

From the monotonicity property (v) we get

\[
\left( a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) - a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, p^k_\varepsilon \right) \right) \cdot \left( \nabla u^\varepsilon (x, t) - p^k_\varepsilon (x, t) \right) \geq 0,
\]

which after integration and expansion takes the form

\[
\int_{\Omega_T} a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \cdot \nabla u^\varepsilon (x, t) - a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \cdot p^k_\varepsilon (x, t) -
\]

\[
a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, p^k_\varepsilon \right) \cdot \nabla u^\varepsilon (x, t) + a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, p^k_\varepsilon \right) \cdot p^k_\varepsilon (x, t) \ dx dt \geq 0.
\]

Replacing \( u_c \) with \( u^\varepsilon \) in (102), we get an alternative way of expressing the first term and the above inequality can be written as

\[
\int_{\Omega_T} f (x, t) u^\varepsilon (x, t) - a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \cdot p^k_\varepsilon (x, t) -
\]

\[
a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, p^k_\varepsilon \right) \cdot \nabla u^\varepsilon (x, t) + a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, p^k_\varepsilon \right) \cdot p^k_\varepsilon (x, t) \ dx dt -
\]

\[
\int_0^T \langle \partial_t u^\varepsilon (x, t), u^\varepsilon (x, t) \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \ dx dt \geq 0.
\]
We recall that \( a(y_1, y_2, s, p^k) \), \( p^k \) and their product are admissible test functions and get, up to a subsequence, that

\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} f(x, t) u(x, t) - a^0(x, t, y_1, y_2, s) \cdot p^k(x, t, y_1, y_2, s) - a(y_1, y_2, s, p^k) \cdot (\nabla u(x, t) + \nabla y_1 u_1(x, t, y_1, s) + \nabla y_2 u_2(x, t, y_1, y_2, s)) + a(y_1, y_2, s, p^k) \cdot p^k(x, t, y_1, y_2, s) \, dy_2 dy_1 ds dx dt
\]

when \( \varepsilon \) tends to zero. Our next step is to let \( k \) tend to infinity. We have

\[
p^k(x, t, y_1, y_2, s) \to \nabla u(x, t) + \nabla y_1 u_1(x, t, y_1, s) + \nabla y_2 u_2(x, t, y_1, y_2, s) + \delta c(x, t, y_1, y_2, s)
\]

in \( L^2(\Omega_T \times \mathcal{Y}_{2,1})^N \). Moreover,

\[
a(y_1, y_2, s, p^k) \to a(y_1, y_2, s, \nabla u + \nabla y_1 u_1 + \nabla y_2 u_2 + \delta c)
\]

and

\[
a(y_1, y_2, s, p^k) \cdot p^k(x, t, y_1, y_2, s) \to a(y_1, y_2, s, \nabla u + \nabla y_1 u_1 + \nabla y_2 u_2 + \delta c) \cdot (\nabla u(x, t) + \nabla y_1 u_1(x, t, y_1, s) + \nabla y_2 u_1(x, t, y_1, y_2, s) + \delta c(x, t, y_1, y_2, s))
\]

almost everywhere in \( \Omega_T \times \mathcal{Y}_{2,1} \). The condition (iv) yields that

\[
|a(y_1, y_2, s, p^k)| \leq \beta \left( 1 + |p^k(x, t, y_1, y_2, s)| \right),
\]

which, together with the Cauchy-Schwarz inequality, gives us

\[
|a(y_1, y_2, s, p^k) \cdot p^k(x, t, y_1, y_2, s)| \leq |a(y_1, y_2, s, p^k)| \cdot |p^k(x, t, y_1, y_2, s)| \leq \beta \left( 1 + |p^k(x, t, y_1, y_2, s)| \right) |p^k(x, t, y_1, y_2, s)| = \beta \left( |p^k(x, t, y_1, y_2, s)| + |p^k(x, t, y_1, y_2, s)|^2 \right).
\]

For \( k \to \infty \), it holds that

\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} |p^k(x, t, y_1, y_2, s)| + |p^k(x, t, y_1, y_2, s)|^2 \, dy_2 dy_1 ds dx dt \\
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} |\nabla u(x, t) + \nabla y_1 u_1(x, t, y_1, s) + \nabla y_2 u_2(x, t, y_1, y_2, s) + \delta c(x, t, y_1, y_2, s)| +
\]

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Lebesgue’s generalized majorized convergence theorem now gives that
\[
\left| \nabla u (x, t) + \nabla_{y_1} u_1 (x, t, y_1, y_2, s) + \nabla_{y_2} u_2 (x, t, y_1, y_2, s) + \delta c (x, t, y_1, y_2, s) \right|^2 \ dy_2 dy_1 ds dx dt.
\]

We have left:
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} a \left( y_1, y_2, s, p^k \right) \cdot p^k (x, t, y_1, y_2, s) \ dy_2 dy_1 ds dx dt \rightarrow
\]
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} a (y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2 + \delta c) \cdot (\nabla u (x, t) + \nabla_{y_1} u_1 (x, t, y_1, s) + \nabla_{y_2} u_2 (x, t, y_1, y_2, s) + \delta c (x, t, y_1, y_2, s)) \ dy_2 dy_1 ds dx dt.
\]

We conclude that when \( k \) tends to infinity (110) turns into
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} f (x, t) u (x, t) - a^0 (x, t, y_1, y_2, s) \cdot (\nabla u (x, t) + \nabla_{y_1} u_1 (x, t, y_1, s) + \nabla_{y_2} u_2 (x, t, y_1, y_2, s) + \delta c (x, t, y_1, y_2, s)) - a (y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2 + \delta c) \cdot (\nabla u (x, t) + \nabla_{y_1} u_1 (x, t, y_1, s) + \nabla_{y_2} u_2 (x, t, y_1, y_2, s) + \delta c (x, t, y_1, y_2, s)) \ dy_2 dy_1 ds dx dt -
\]
\[
\int_0^T \left< \partial_t u (x, t), u (x, t) \right>_{H^{-1} (\Omega), H^1_0 (\Omega)} \ dx dt \geq 0.
\]

Some of the terms cancel out and, using (106) with \( v_c \) replaced by \( u \), we get
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} -a^0 (x, t, y_1, y_2, s) \cdot \nabla_{y_1} u_1 (x, t, y_1, s) - a^0 (x, t, y_1, y_2, s) \cdot \nabla_{y_2} u_2 (x, t, y_1, y_2, s) - a^0 (x, t, y_1, y_2, s) \cdot \delta c (x, t, y_1, y_2, s) + a (y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2 + \delta c) \cdot \delta c (x, t, y_1, y_2, s) \ dy_2 dy_1 ds dx dt \geq 0.
\]

Here, the first and second term vanish due to (108) and (107), respectively. We have left:
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} (-a^0 (x, t, y_1, y_2, s) + a (y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2 + \delta c)) \cdot \delta c (x, t, y_1, y_2, s) \ dy_2 dy_1 ds dx dt \geq 0.
\]
Dividing by $\delta$ and letting $\delta \to 0$, we get
\[
a^0 (x, t, y_1, y_2, s) = a (y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) .
\]

By the uniqueness of $u$ the whole sequence converges and the proof is complete. ■

Now we proceed with the case when $r = 2$. One might wonder whether there is any reason to believe that this case would be different from the previous one; that is, if we can expect to get other local problems. In the case with one micro-scale in space and time, respectively, we get a local problem which is parabolic when $r = 2$; see e.g. [DaMu], [Hol1], [Hol2], [Sva2] and [Wel]. Hence, it is not unreasonable to assume that the situation would be different for this choice of $r$ compared to the case in Theorem 72.

**Theorem 73** Let $\{u^\varepsilon\}$ be a sequence of solutions in $H^1 (0, T; H^1_0 (\Omega), H^{-1} (\Omega))$ to (101) for $r = 2$. Then
\[
\begin{align*}
    u^\varepsilon (x, t) & \to u (x, t) \quad \text{in } L^2 (\Omega_T), \\
    u^\varepsilon (x, t) & \to u (x, t) \quad \text{in } L^2 (0, T; H^1_0 (\Omega))
\end{align*}
\]
and
\[
\nabla u^\varepsilon (x, t) = \frac{3}{2} \nabla u (x, t) + \nabla_{y_1} u_1 (x, t, y_1, s) + \nabla_{y_2} u_2 (x, t, y_1, y_2, s),
\]
where $u \in H^1 (0, T; H^1_0 (\Omega), H^{-1} (\Omega))$, $u_1 \in L^2 (\Omega_T \times (0, 1); H^1_2 (Y_1) / \mathbb{R})$ and $u_2 \in L^2 (\Omega_T \times Y_1; H^1 (Y_2) / \mathbb{R})$, and where $u$ is the unique solution to the homogenized problem
\[
\begin{align*}
\partial_t u (x, t) - \nabla \cdot b (x, t, \nabla u) &= f (x, t) \quad \text{in } \Omega_T, \\
    u (x, 0) &= u^0 (x) \quad \text{in } \Omega, \\
    u (x, t) &= 0 \quad \text{on } \partial \Omega \times (0, T).
\end{align*}
\]

Here,
\[
b (x, t, \nabla u) = \int_{Y_2, 1} a (y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) \, dy_2 dy_1 ds,
\]
where $u_1$ and $u_2$ are the unique solutions to the system of local problems
\[
\begin{align*}
    -\nabla_{y_2} \cdot a (y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) &= 0, \\
    \partial_s u_1 (x, t, y_1, s) - \nabla_{y_1} \cdot \int_{Y_2} a (y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) \, dy_2 &= 0.
\end{align*}
\]
Proof. Following the same line of reasoning as in the proof of Theorem 72, we have, up to a subsequence, that

\[
\begin{align*}
\epsilon u(x, t) & \to u(x, t) \quad \text{in } L^2(\Omega_T), \\
\epsilon v(x, t) & \to v(x, t) \quad \text{in } L^2(0, T; H^1_0(\Omega))
\end{align*}
\]

and

\[
\nabla u^\epsilon(x, t) \mathcal{L} \nabla u(x, t) + \nabla_y u_1(x, t, y_1, s) + \nabla_y u_2(x, t, y_1, y_2, s),
\]

where \( u \in L^2(0, T; H^1_0(\Omega)), u_1 \in L^2(\Omega_T \times (0, 1); H^1_0(Y_1)/\mathbb{R}) \) and \( u_2 \in L^2(\Omega_T \times Y_{1,1}; H^1_0(Y_2)/\mathbb{R}) \). In addition we get that, up to a subsequence,

\[
a \left( \frac{x}{\epsilon^2}, \frac{x}{\epsilon^2}, \frac{t}{\epsilon^2}, \nabla u^\epsilon \right) \mathcal{L} \left( \frac{x}{\epsilon^2}, \frac{x}{\epsilon^2}, \frac{t}{\epsilon^2}, \nabla u \right) = a^0(x, t, y_1, y_2, s)
\]

for some \( a^0 \in L^2(\Omega_T \times Y_{2,1})^N \). Furthermore, we know that if we choose \( v \in H^1_0(\Omega) \) and \( c \in D(0, T) \) in (102), we end up with the limit (106); that is,

\[
b(x, t, \nabla u) = \int_{Y_{2,1}} a^0(x, t, y_1, y_2, s) \, dy_2 dy_1 ds.
\]

To find the functions \( u_1 \) and \( u_2 \) we need local problems. In order to find the first local problem, we choose the test functions

\[
v(x) = \epsilon^2 v_1(x) v_2 \left( \frac{x}{\epsilon} \right) v_3 \left( \frac{x}{\epsilon^2} \right), \quad c(t) = c_1(t) c_2 \left( \frac{t}{\epsilon^2} \right),
\]

in (102), where \( v_1 \in D(\Omega), v_2 \in C^\infty_\sharp(Y_1), v_3 \in C^\infty_\sharp(Y_2)/\mathbb{R}, c_1 \in D(0, T) \) and \( c_2 \in C^\infty_\sharp(0, 1) \). Carrying out the differentiations yields

\[
\int_{\Omega_T} \frac{\epsilon^2}{\epsilon^2} v_1(x) v_2 \left( \frac{x}{\epsilon} \right) v_3 \left( \frac{x}{\epsilon^2} \right) \left( \frac{\epsilon^2}{\epsilon^2} \partial_t c_1(t) c_2 \left( \frac{t}{\epsilon^2} \right) + c_1(t) \partial_s c_2 \left( \frac{t}{\epsilon^2} \right) \right) + a \left( \frac{x}{\epsilon^2}, \frac{x}{\epsilon^2}, \frac{t}{\epsilon^2}, \nabla u^\epsilon \right) \cdot \left( \frac{\epsilon^2}{\epsilon^2} \nabla v_1(x) v_2 \left( \frac{x}{\epsilon} \right) v_3 \left( \frac{x}{\epsilon^2} \right) \right) + \epsilon v_1(x) \nabla y_1 v_2 \left( \frac{x}{\epsilon} \right) v_3 \left( \frac{x}{\epsilon^2} \right) + v_1(x) v_2 \left( \frac{x}{\epsilon} \right) \nabla y_2 v_3 \left( \frac{x}{\epsilon^2} \right) c_1(t) c_2 \left( \frac{t}{\epsilon^2} \right) \, dx dt =
\int_{\Omega_T} f(x, t) \epsilon^2 v_1(x) v_2 \left( \frac{x}{\epsilon} \right) v_3 \left( \frac{x}{\epsilon^2} \right) c_1(t) c_2 \left( \frac{t}{\epsilon^2} \right) \, dx dt
\]
and letting $\varepsilon$ pass to zero we get
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} - u(x, t) v_1(x) v_2(y_1) v_3(y_2) c_1(t) \partial_s c_2(s) + \quad a^0(x, t, y_1, y_2, s) \cdot v_1(x) v_2(y_1) \nabla_{y_2} v_3(y_2) c_1(t) c_2(s) \ dy_2 dy_1 ds dx dt = 0.
\]

The first term vanishes since $c_2$ is periodic. Applying the variational lemma to the remaining part, we obtain
\[
\int_{\mathcal{Y}_2} a^0(x, t, y_1, y_2, s) \cdot \nabla_{y_2} v_3(y_2) \ dy_2 = 0 \tag{115}
\]
for all $v_3 \in H^1_{\Omega} (Y_2) / \mathbb{R}$, which will turn out to be the weak form of (113).

To find the next local problem, we choose the test functions
\[
v(x) = \varepsilon v_1(x) v_2 \left( \frac{x}{\varepsilon} \right), \quad c(t) = c_1(t) c_2 \left( \frac{t}{\varepsilon^2} \right) \tag{116}
\]
in (102) as well as in (106), where $v_1 \in D(\Omega)$, $v_2 \in C^\infty(\Omega_1) / \mathbb{R}$, $c_1 \in D(0, T)$ and $c_2 \in C^\infty(0, 1)$. For the difference between these two equations, we get
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} \frac{1}{\varepsilon} (u^\varepsilon(x, t) - u(x, t)) v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) \cdot
\left( \varepsilon^2 \partial_t c_1(t) c_2 \left( \frac{t}{\varepsilon^2} \right) + c_1(t) \partial_s c_2 \left( \frac{t}{\varepsilon^2} \right) \right) +
\left( \int_{\mathcal{Y}_{2,1}} a^0(x, t, y_1, y_2, s) \ dy_2 dy_1 ds - a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}, \nabla u^\varepsilon \right) \right) \cdot
\left( \varepsilon \nabla v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) + v_1(x) \nabla_{y_1} v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1(t) c_2 \left( \frac{t}{\varepsilon^2} \right) \ dx dt = 0.
\]

Applying Theorem 63 and 3,2-scale convergence yields
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{1,1}} u_1(x, t, y_1, s) v_1(x) v_2(y_1) c_1(t) \partial_s c_2(s) \ dy_1 ds dx dt +
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} \left( \int_{\mathcal{Y}_{2,1}} a^0(x, t, y_1, y_2, s) \ dy_2 dy_1 ds \right) \cdot
\left( v_1(x) \nabla_{y_1} v_2(y_1) c_1(t) c_2(s) \ dy_2 dy_1 ds dx dt -
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} a^0(x, t, y_1, y_2, s) \cdot v_1(x) \nabla_{y_1} v_2(y_1) c_1(t) c_2(s) \ dy_2 dy_1 ds dx dt = 0
\]
as $\varepsilon$ tends to zero. Due to the periodicity of $v_2$, the middle term vanishes and we get

$$\int_{\Omega_T} \left( \int_{Y_{1,1}} u_1(x,t,y_1,s) v_2(y_1) \partial_s c_2(s) - \left( \int_{Y_2} a^0(x,t,y_1,y_2,s) \, dy_2 \right) \cdot \nabla y_1 v_2(y_1) c_2(s) \, dy_1 ds \right) \, v_1(x) \, c_1(t) \, dxdt = 0.$$  

By the variational lemma

$$\int_{Y_{1,1}} u_1(x,t,y_1,s) v_2(y_1) \partial_s c_2(s) - \left( \int_{Y_2} a^0(x,t,y_1,y_2,s) \, dy_2 \right) \cdot \nabla y_1 v_2(y_1) c_2(s) \, dy_1 ds = 0$$  

for all $v_2 \in H^1_\#(Y_1)/\mathbb{R}, c_2 \in C^\infty(0,1)$ and almost everywhere in $\Omega_T$, i.e. we have the weak form of (114) if we can prove that

$$a^0(x,t,y_1,y_2,s) = a(y_1,y_2,s,\nabla u + \nabla y_1 u_1 + \nabla y_2 u_2).$$

For the characterization of the limit $a^0$, we use the method with perturbed test functions carried out in detail for the case when $0 < r < 2$. As in that case we arrive at (111), i.e.

$$\int_{\Omega_T} \int_{Y_{2,1}} -a^0(x,t,y_1,y_2,s) \cdot \nabla y_1 u_1(x,t,y_1,s) - a^0(x,t,y_1,y_2,s) \cdot \nabla y_2 u_2(x,t,y_1,y_2,s) - a^0(x,t,y_1,y_2,s) \cdot \delta c(x,t,y_1,y_2,s) + a(y_1,y_2,s,\nabla u + \nabla y_1 u_1 + \nabla y_2 u_2 + \delta c) \cdot \delta c(x,t,y_1,y_2,s) dy_2 dy_1 ds dx dt \geq 0.$$

From (117) it follows that we may replace $a^0 \cdot \nabla y_1$ in (118) with $\partial_s u_1$ and since $\partial_s u_1(x,t,\cdot,\cdot) \in L^2_\#(0,1;(H^1_\#(Y_1)/\mathbb{R})')$ (c.f. Definition 3.1, Lemma 3.4 and Section 4.2 in [NgWo2]) apply Proposition 64. Hence, the first term in (118) vanishes. By (115), so does the second term. Dividing by $\delta$ and letting $\delta \to 0$, we have

$$a^0(x,t,y_1,y_2,s) = a(y_1,y_2,s,\nabla u + \nabla y_1 u_1 + \nabla y_2 u_2).$$

This means that (115) and (117) are the weak formulations of the local problems (113) and (114), respectively, which completes the proof. ■
Our next concern is to study the homogenization procedure for (101) when \(2 < r < 3\). Again, looking at the case with one spatial and one temporal micro-scale, we observe that for \(r > 2\) the local time variable is averaged away in one of the local problems. It turns out that this is the case also for our problem, as we can see in the next theorem.

**Theorem 74**  Let \(\{u^\varepsilon\}\) be a sequence of solutions in \(H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega))\) to (101) for \(2 < r < 3\). Then it holds that
\[
u^\varepsilon(x, t) \to u(x, t) \quad \text{in} \quad L^2(\Omega_T),
\]
\[
u^\varepsilon(x, t) \to u(x, t) \quad \text{in} \quad L^2(0, T; H^1_0(\Omega))
\]
and
\[
\nabla u^\varepsilon(x, t) \overset{3,2}{\to} \nabla u(x, t) + \nabla_{y_1} u_1(x, t, y_1) + \nabla_{y_2} u_2(x, t, y_1, y_2, s),
\]
where \(u \in H^1(0, T; H^1_0(\Omega), H^{-1}(\Omega))\) is the unique solution to the homogenized problem
\[
\partial_t u(x, t) - \nabla \cdot b(x, t, \nabla u) = f(x, t) \quad \text{in} \quad \Omega_T,
\]
\[
u(x, 0) = u^0(x) \quad \text{in} \quad \Omega,
\]
\[
u(x, t) = 0 \quad \text{on} \partial \Omega \times (0, T),
\]
where in turn
\[
b(x, t, \nabla u) = \int_{Y_2} a(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) \, dy_2 dy_1 ds.
\]
Here, \(u_1 \in L^2(\Omega_T; H^1_0(Y_1)/\mathbb{R})\) and \(u_2 \in L^2(\Omega_T \times Y_{1,1}; H^1_0(Y_2)/\mathbb{R})\) are the unique solutions to the system of local problems
\[
-\nabla_{y_2} \cdot a(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) = 0, \quad (119)
\]
\[
\partial_s u_1(x, t, y_1, s) = 0, \quad (120)
\]
\[
-\nabla_{y_1} \cdot \int_0^1 \int_{Y_2} a(y_1, y_2, s, \nabla u + \nabla_{y_1} u_1 + \nabla_{y_2} u_2) \, dy_2 ds = 0. \quad (121)
\]

**Proof.** As in the proof of Theorem 72, we know that, up to a subsequence,
\[
u^\varepsilon(x, t) \to u(x, t) \quad \text{in} \quad L^2(\Omega_T),
\]
\[
u^\varepsilon(x, t) \to u(x, t) \quad \text{in} \quad L^2(0, T; H^1_0(\Omega))
\]
and

\[ \nabla u^\varepsilon(x, t) \overset{3,2}{\to} \nabla u(x, t) + \nabla y_1 u_1(x, t, y_1, s) + \nabla y_2 u_2(x, t, y_1, y_2, s), \]

where \( u \in L^2(0, T; H^1_0(\Omega)) \), \( u_1 \in L^2(\Omega_T \times (0, 1); H^1_1(Y_1)/\mathbb{R}) \) and \( u_2 \in L^2(\Omega_T \times Y_{1,1}; H^1_1(Y_2)/\mathbb{R}) \). Moreover, up to a subsequence

\[ a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \overset{3,2}{\to} a^0(x, t, y_1, y_2, s) \]

for some \( a^0 \in L^2(\Omega_T \times Y_{2,1})^N \). Also, we know that if we choose \( v \in H^1_0(\Omega) \) and \( c \in D(0, T) \) in (102), we end up with the limit (106) and hence

\[ b(x, t, \nabla u) = \int_{Y_{2,1}} a^0(x, t, y_1, y_2, s) \ dy_2 dy_1 ds. \]

To find the first local problem, we study the difference between (102) and (106) for the test functions

\[ v(x) = \varepsilon^2 v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right), \quad c(t) = c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right), \]

where \( v_1 \in D(\Omega), v_2 \in C_\infty(Y_1), v_3 \in C_\infty(Y_2)/\mathbb{R}, c_1 \in D(0, T) \) and \( c_2 \in C_\infty(0, 1) \). We get

\[
\int_{\Omega_T} \left( u^\varepsilon(x, t) - u(x, t) \right) \varepsilon^2 v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) \cdot \\
\left( \partial_t c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) + \varepsilon^{-r} c_1(t) \partial_s c_2 \left( \frac{t}{\varepsilon^r} \right) \right) + \\
\left( \int_{Y_{2,1}} a^0(x, t, y_1, y_2, s) \ dy_2 dy_1 ds - a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \right) \cdot \\
\left( \varepsilon^2 \nabla v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) + \varepsilon v_1(x) \nabla y_1 v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) + \\
v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) \nabla y_2 v_3 \left( \frac{x}{\varepsilon^2} \right) \right) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \ dx dt = 0,
\]

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and hence
\[
\int_{\Omega_T} \left( \frac{1}{\varepsilon} (u^\varepsilon (x, t) - u (x, t)) v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) \right) \cdot \\
\left( \varepsilon^3 \partial_t c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) + \varepsilon^{3-r} c_1 (t) \partial_x c_2 \left( \frac{t}{\varepsilon^r} \right) \right) + \\
\left( \int_{\mathcal{Y}_{2,1}} a^0 (x, t, y_1, y_2, s) \ dy_2 dy_1 ds - a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \right) \cdot \\
\left( \varepsilon^2 \nabla v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) + \varepsilon v_1 (x) \nabla y_1 v_2 \left( \frac{x}{\varepsilon} \right) v_3 \left( \frac{x}{\varepsilon^2} \right) + \\
v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) \nabla y_2 v_3 \left( \frac{x}{\varepsilon^2} \right) c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) \ dx dt = 0.
\]

When \( \varepsilon \) tends to zero, we obtain
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} \left( \int_{\mathcal{Y}_{2,1}} a^0 (x, t, y_1, y_2, s) \ dy_2 dy_1 ds - a^0 (x, t, y_1, y_2, s) \right) \cdot \\
v_1 (x) v_2 (y_1) \nabla y_2 v_3 (y_2) c_1 (t) c_2 (s) \ dy_2 dy_1 ds dx dt = 0
\]
and since \( v_3 \) is \( Y_2 \)-periodic, the first term vanishes and we obtain
\[
\int_{\Omega_T} \int_{\mathcal{Y}_{2,1}} a^0 (x, t, y_1, y_2, s) \cdot v_1 (x) v_2 (y_1) \nabla y_2 v_3 (y_2) c_1 (t) c_2 (s) \ dy_2 dy_1 ds dx dt = 0.
\]
Due to the variational lemma,
\[
\int_{\mathcal{Y}_2} a^0 (x, t, y_1, y_2, s) \cdot \nabla y_2 v_3 (y_2) \ dy_2 = 0
\tag{122}
\]
for all \( v_3 \in H^1 \_2 (Y_2) / \mathbb{R} \).

Our next concern is the second local problem. We now study the difference between (102) and (106) for the test functions
\[
v (x) = \varepsilon^{r-1} v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right), \quad c (t) = c_1 (t) c_2 \left( \frac{t}{\varepsilon^r} \right),
\]
where \( v_1 \in D(\Omega), \ v_2 \in C^\infty_2 (Y_1) / \mathbb{R}, \ c_1 \in D (0, T) \) and \( c_2 \in C^\infty_2 (0, 1) \), and
arrive at

\[
\int_{\Omega_T} \left( u^\varepsilon (x, t) - u(x, t) \right) \varepsilon^{-1} v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) \partial_t \left( c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \right) + \\
\left( \int_{\mathcal{Y}_{2,1}} a^0(x, t, y_1, y_2, s) \ dy_2 dy_1 ds - a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \right) .
\]

\[
\nabla \left( \varepsilon^{-1} v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \ dx dt = 0.
\]

Differentiating and rewriting, we have

\[
\int_{\Omega_T} \frac{1}{\varepsilon} \left( u^\varepsilon (x, t) - u(x, t) \right) v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) \cdot \\
\left( \varepsilon^r \partial_t c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) + c_1(t) \partial_s c_2 \left( \frac{t}{\varepsilon^r} \right) \right) + \\
\left( \int_{\mathcal{Y}_{2,1}} a^0(x, t, y_1, y_2, s) \ dy_2 dy_1 ds - a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \right) .
\]

\[
\left( \varepsilon^{r-1} \nabla v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) + \varepsilon^{r-2} v_1(x) \nabla y_1 v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1(t) c_2 \left( \frac{t}{\varepsilon^r} \right) \ dx dt = 0
\]

and for \(2 < r < 3\), by Theorem 63, as \(\varepsilon\) tends to zero, we obtain that

\[
\int_{\Omega_T} \int_{\mathcal{Y}_{1,1}} u_1(x, t, y_1, s) v_1(x) v_2(y_1) c_1(t) \partial_s c_2(s) \ dy_1 ds dx dt = 0.
\]

Finally, applying the variational lemma we get

\[
\int_{\mathcal{Y}_{1,1}} u_1(x, t, y_1, s) v_2(y_1) \partial_s c_2(s) \ dy_1 ds = 0 \tag{123}
\]

for all \(v_2 \in H^1_{\mathbb{Z}}(Y_1) / \mathbb{R}\) and \(c_2 \in C^\infty_c(0, 1)\), i.e. we have the weak form of (120). We deduce that \(u_1\) does not depend on the local time variable \(s\).

To find the third local problem, we now choose the test functions

\[
v(x) = \varepsilon v_1(x) v_2 \left( \frac{x}{\varepsilon} \right), \ c(t) = c_1(t)
\]
in (102), where $v_1 \in D(\Omega)$, $v_2 \in C^\infty_0(Y_1)/\mathbb{R}$ and $c_1 \in D(0,T)$. After differentiations, we get

$$
\int_{\Omega_T} -u^\varepsilon (x,t) \varepsilon v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) \partial_t c_1 (t) + a \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \frac{t}{\varepsilon^r}, \nabla u^\varepsilon \right) \cdot 
\left( \varepsilon \nabla v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) + v_1 (x) \nabla_y v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1 (t) \ dx dt =
\int_{\Omega_T} f (x,t) \varepsilon v_1 (x) v_2 \left( \frac{x}{\varepsilon} \right) c_1 (t) \ dx dt.
$$

When $\varepsilon$ tends to zero, we obtain

$$
\int_{\Omega_T} \int_{Y_2,1} a^0 (x,t,y_1,y_2,s) \cdot v_1 (x) \nabla_y v_2 (y_1) c_1 (t) \ dy_2 dy_1 ds dx dt = 0.
$$

By applying the variational lemma, we get

$$
\int_{Y_1} \left( \int_0^1 \int_{Y_2} a^0 (x,t,y_1,y_2,s) \ dy_2 ds \right) \cdot \nabla_y v_2 (y_1) \ dy_1 = 0 \quad (124)
$$

for all $v_2 \in H^1_\sharp (Y_1)/\mathbb{R}$.

For characterization of the limit $a^0$, we again use the method with perturbed test functions carried out in detail for the case when $0 < r < 2$, and deduce that

$$
a^0 (x,t,y_1,y_2,s) = a \left( y_1, y_2, s, \nabla u + \nabla_y u_1 + \nabla_y u_2 \right).
$$

This means that (122) and (124) are the weak formulations of the local problems (119) and (121), respectively. ■

**Remark 75** Note that with the multiscale convergence method used in this chapter, the existence of the solution to the local problems is immediate since the functions $u$, $u_1$ and $u_2$ appear as a multiscale limit and constituents of such limits of $\{u^\varepsilon\}$, respectively. Methods to prove existence and uniqueness for solutions to local problems of the kind found above can be concluded from the considerations in Sections 4.2.1 and 4.2.3 in [Pan].
5.3.3 Homogenization of some linear parabolic operators with two spatial and three temporal scales

In this section, we study the homogenization of the parabolic equation

\[
\partial_t u^\varepsilon (x,t) - \nabla \cdot \left( a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u^\varepsilon (x,t) \right) = f(x,t) \quad \text{in } \Omega \times (0, T),
\]

\[
u^\varepsilon (x,0) = u^0(x) \quad \text{in } \Omega, \quad \text{(125)}
\]

\[
u^\varepsilon (x,t) = 0 \quad \text{on } \partial \Omega \times (0, T),
\]

where \( f \in L^2(\Omega_T) \), \( u^0 \in L^2(\Omega) \) and

\[
a : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}^{N \times N}
\]

is assumed to satisfy the following conditions, where \( 0 < \alpha \leq \beta \):

(i) \( a \in L^\infty_\ast (\mathcal{Y}_{1,2})^{N \times N} \).

(ii) \( |a(y, s_1, s_2) \xi| \leq \beta |\xi| \) for all \( (y, s_1, s_2) \in \mathbb{R}^N \times \mathbb{R}^2 \) and all \( \xi \in \mathbb{R}^N \).

(iii) \( a(y, s_1, s_2) \xi \cdot \xi \geq \alpha |\xi|^2 \) for all \( (y, s_1, s_2) \in \mathbb{R}^N \times \mathbb{R}^2 \) and all \( \xi \in \mathbb{R}^N \).

As we noted in Remark 9, this problem allows, for any fixed \( \varepsilon > 0 \), a unique solution \( u^\varepsilon \in H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \) if (i) and (iii) are fulfilled. This means that there exists a unique function \( u^\varepsilon \in H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega)) \) such that \( u^\varepsilon (x,0) = u^0(x) \) and

\[
\int_{\Omega_T} -u^\varepsilon (x,t) v(x) \partial_t c(t) + a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon^2} \right) \nabla u^\varepsilon (x,t) \cdot \nabla v(x) c(t) \, dxdt =
\]

\[
\int_{\Omega_T} f(x,t) v(x) c(t) \, dxdt \quad \text{(126)}
\]

for all \( v \in H_0^1(\Omega) \) and \( c \in D(0,T) \). Under the assumptions above, we also have

\[
\|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C
\]

and

\[
\|u^\varepsilon\|_{H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))} \leq C \quad \text{(127)}
\]

for some positive constant \( C \); see Propositions 23.23 and 23.30 in [ZeiIIA] or Theorems 11.2 and 11.4 in [CiDo].
Furthermore, \( a \) belongs to \( \mathcal{M}(\alpha, \beta, \mathbb{R}^N \times \mathbb{R}^2) \). From this, it can be seen that 
\[
\{a(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \frac{y}{\varepsilon})\}
\]
satisfies the conditions for \( G \)-convergence for linear parabolic problems, i.e. it belongs to \( \mathcal{M}(\alpha, \beta, \Omega_T) \), which we defined in Section 3.2. This means that, at least for a suitable subsequence, there exists a well-posed limit problem of the same type as (125) governed by a coefficient matrix \( b \); see Theorem 25. As in the previous section, our aim is to characterize the \( G \)-limit \( b \) further by means of homogenization procedures.

**Theorem 76** Let \( \{u^\varepsilon\} \) be a sequence of solutions in \( H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega)) \) to (125). Then 
\[
\begin{align*}
&\quad \quad u^\varepsilon(x,t) \rightarrow u(x,t) \quad \text{in} \quad L^2(\Omega_T),
\quad \quad u^\varepsilon(x,t) \rightarrow u(x,t) \quad \text{in} \quad L^2(0,T;H^1_0(\Omega))
\end{align*}
\]
and
\[
\nabla u^\varepsilon(x,t) \overset{2,3}{\rightarrow} \nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2),
\]
where \( u \in H^1(0,T;H^1_0(\Omega),H^{-1}(\Omega)) \) and \( u_1 \in L^2(\Omega_T \times (0,1)^2;H^1_0(Y)/\mathbb{R}) \).

Furthermore, \( u \) is the unique solution to the homogenized problem
\[
\begin{align*}
\partial_t u(x,t) - \nabla \cdot (b \nabla u(x,t)) &= f(x,t) \quad \text{in} \quad \Omega_T, \\
u(x,0) &= u^0(x) \quad \text{in} \quad \Omega, \\
u(x,t) &= 0 \quad \text{on} \quad \partial \Omega \times (0,T),
\end{align*}
\]
where
\[
b \nabla u(x,t) = \int_{Y_{1,2}} a(y,s_1,s_2)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2)) \, dyds_2ds_1.
\]
For \( 0 < r < 2, r \neq 1 \), the function \( u_1 \) is determined by the local problem
\[
-\nabla_y \cdot (a(y,s_1,s_2)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2))) = 0, \tag{128}
\]
for \( r = 2 \) by
\[
\partial_{s_2} u_1(x,t,y,s_1,s_2) - \nabla_y \cdot (a(y,s_1,s_2)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2))) = 0, \tag{129}
\]
and for \( r > 2 \) by the system of local problems
\[
\begin{align*}
\partial_{s_2} u_1(x,t,y,s_1,s_2) &= 0, \tag{130} \\
-\nabla_y \cdot \left( \left( \int_0^1 a(y,s_1,s_2) \, ds_2 \right)(\nabla u(x,t) + \nabla_y u_1(x,t,y,s_1)) \right) &= 0. \tag{131}
\end{align*}
\]
Proof. The a priori estimate (127) allows us to apply Theorem 67. Hence, up to a subsequence,

\[ u^\varepsilon(x,t) \to u(x,t) \text{ in } L^2(\Omega_T), \]

\[ u^\varepsilon(x,t) \to u(x,t) \text{ in } L^2(0,T; H^1_0(\Omega)) \]

and

\[ \nabla u^\varepsilon(x,t) \to \nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2), \]

where \( u \in L^2(0,T; H^1_0(\Omega)) \) and \( u_1 \in L^2(\Omega_T \times (0,1)^2; H^1_0(Y)/\mathbb{R}) \). Choosing \( v \in D(\Omega) \) and \( c \in D(0,T) \) to be independent of \( \varepsilon \) in (126) and letting \( \varepsilon \) tend to zero, by Theorem 67 we get

\[
\int_{\Omega_T} -u(x,t) v(x) \partial_t c(t) + \left( \int_{Y_{1,2}} a(y,s_1,s_2) \left( \nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2) \right) dy ds_2 ds_1 \right) \cdot \nabla v(x) c(t) \, dx dt = \int_{\Omega_T} f(x,t) v(x) c(t) \, dx dt. \tag{132}
\]

In order to find the local problems, we study the difference between (126) and (132), i.e.

\[
\int_{\Omega_T} (u^\varepsilon(x,t) - u(x,t)) v(x) \partial_t c(t) + \left( \int_{Y_{1,2}} a(y,s_1,s_2) \left( \nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2) \right) dy ds_2 ds_1 - a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \frac{t}{\varepsilon^r} \right) \nabla u^\varepsilon(x,t) \right) \cdot \nabla v(x) c(t) \, dx dt = 0. \tag{133}
\]

We choose the test functions

\[ v(x) = \varepsilon v_1(x) v_2 \left( \frac{x}{\varepsilon} \right), \quad c(t) = c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) c_3 \left( \frac{t}{\varepsilon^r} \right), \]

where \( v_1 \in D(\Omega), v_2 \in C^\infty_\sharp(Y)/\mathbb{R}, c_1 \in D(0,T) \) and \( c_2, c_3 \in C^\infty_\sharp(0,1) \) in
(133) and get
\[
\int_{\Omega_T} \frac{1}{\varepsilon} (u^\varepsilon (x, t) - u(x, t)) v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) \left( \varepsilon^2 \partial_t c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) c_3 \left( \frac{t}{\varepsilon^r} \right) + \varepsilon c_1(t) \partial_{s_1} c_2 \left( \frac{t}{\varepsilon} \right) c_3 \left( \frac{t}{\varepsilon^r} \right) + \varepsilon^2 - r \varepsilon c_1(t) \partial_{s_2} c_3 \left( \frac{t}{\varepsilon^r} \right) \right) + \
\left( \int_{Y_{1,2}} a(y, s_1, s_2) (\nabla u(x, t) + \nabla_y u_1(x, t, y, s_1, s_2)) dy ds_2 ds_1 - (134) \right) \
\int_{Y_{1,2}} \left( \int a(x, s_1, s_2) (\nabla u(x, t) + \nabla_y u_1(x, t, y, s_1, s_2)) dy ds_2 ds_1 \right) \cdot \nabla u^\varepsilon(x, t) \
\cdot \left( \varepsilon \nabla v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) + v_1(x) \nabla_y v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) c_3 \left( \frac{t}{\varepsilon^r} \right) dx dt = 0.
\]

Next, we let \( \varepsilon \) pass to zero.

For the case when \( 0 < r < 2 \) we have, using Theorems 68 and 67 for the first and second term above, respectively,
\[
\int_{\Omega_T} \int_{Y_{1,2}} \left( \int_{Y_{1,2}} a(y, s_1, s_2) (\nabla u(x, t) + \nabla_y u_1(x, t, y, s_1, s_2)) dy ds_2 ds_1 - a(y, s_1, s_2) (\nabla u(x, t) + \nabla_y u_1(x, t, y, s_1, s_2)) \cdot v_1(x) \nabla_y v_2(y) c_1(t) c_2(s_1) c_3(s_2) dy ds_2 ds_1 dx dt = 0
\]
and due to the periodicity of \( v_2 \), we obtain
\[
\int_{\Omega_T} \int_{Y_{1,2}} -a(y, s_1, s_2) (\nabla u(x, t) + \nabla_y u_1(x, t, y, s_1, s_2)) \cdot v_1(x) \nabla_y v_2(y) c_1(t) c_2(s_1) c_3(s_2) dy ds_2 ds_1 dx dt = 0.
\]

By the variational lemma,
\[
\int_Y a(y, s_1, s_2) (\nabla u(x, t) + \nabla_y u_1(x, t, y, s_1, s_2)) \cdot \nabla_y v_2(y) \ dy = 0
\]
for all \( v_2 \in C_0^\infty(Y) / \mathbb{R} \), and hence, by density, for all \( v_2 \in H_0^1(Y) / \mathbb{R} \), a.e. in \( \Omega_T \times (0, 1)^2 \). This is the weak form of (128).
For \( r = 2 \), according to Theorems 67 and 68, (134) approaches

\[
\int_{\Omega_T} \int_{\mathcal{Y}_{1,2}} u_1(x, t, y, s_1, s_2) v_1(x) v_2(y) c_1(t) c_2(s_1) \partial_{s_2} c_3(s_2) \ dy ds_2 ds_1 dx dt + \\
\int_{\Omega_T} \int_{\mathcal{Y}_{1,2}} \left( \int_{\mathcal{Y}_{1,2}} a(y, s_1, s_2) (\nabla u(x, t) + \nabla_y u_1(x, t, y, s_1, s_2)) \ dy ds_2 ds_1 \right) \cdot \vspace{1em}
\vspace{1em} v_1(x) \nabla_y v_2(y) c_1(t) c_2(s_1) c_3(s_2) - a(y, s_1, s_2) (\nabla u(x, t) + \nabla_y u_1(x, t, y, s_1, s_2)) \cdot \vspace{1em}
\vspace{1em} v_1(x) \nabla_y v_2(y) c_1(t) c_2(s_1) c_3(s_2) \ dy ds_2 ds_1 dx dt = 0,
\]

when \( \varepsilon \) tends to zero. Since \( v_2 \) is periodic, the middle term vanishes and we have

\[
\int_{\mathcal{Y}_{1,1}} u_1(x, t, y, s_1, s_2) v_1(x) v_2(y) c_1(t) c_2(s_1) \partial_{s_2} c_3(s_2) - \\
\int_{\mathcal{Y}_{1,1}} v_1(x) \nabla_y v_2(y) c_1(t) c_2(s_1) c_3(s_2) \ dy ds_2 = 0,
\]

for all \( v_2 \in H^1_T(Y)/\mathbb{R} \) and all \( c_3 \in C_4^\infty(0, 1) \), a.e. in \( \Omega_T \times (0, 1) \). We have found the weak form of (129).

For the case when \( r > 2 \), we choose the test functions

\[
v(x) = \varepsilon^{-1} v_1(x) v_2 \left( \frac{x}{\varepsilon} \right), \quad c(t) = c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) c_3 \left( \frac{t}{\varepsilon^r} \right),
\]

in the difference (133), where \( v_1 \in D(\Omega), v_2 \in C_4^\infty(Y)/\mathbb{R}, c_1 \in D(0, T) \) and \( c_2, c_3 \in C_4^\infty(0, 1) \) and obtain

\[
\int_{\Omega_T} \frac{1}{\varepsilon} (w(x, t) - u(x, t)) v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) \left( \varepsilon^r \partial_t c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) c_3 \left( \frac{t}{\varepsilon^r} \right) + \\
\varepsilon^{r-1} c_1(t) \partial_{s_1} c_2 \left( \frac{t}{\varepsilon} \right) c_3 \left( \frac{t}{\varepsilon^r} \right) + c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) \partial_{s_2} c_3 \left( \frac{t}{\varepsilon^r} \right) + \\
\because \right) \right).
\]

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\[
\left( \int_{\mathcal{Y}_1,2} a(y, s_1, s_2)(\nabla u(x, t) + \nabla_y u_1(x, t, y, s_1, s_2)) \, dy ds_2 ds_1 - \right.
\]
\[
a \left( \frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \frac{t}{\varepsilon_r} \right) \nabla u^\varepsilon(x, t) \right) \cdot \left( (\varepsilon^{-1} \nabla v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) + \varepsilon^{-2} v_1(x) \nabla_y v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) c_3 \left( \frac{t}{\varepsilon_r} \right) \, dx dt = 0.
\]

When \(\varepsilon\) passes to zero, we get according to Theorem 68
\[
\int_{\mathcal{Y}_1,2} u_1(x, t, y, s_1, s_2) v_1(x) v_2(y) c_1(t) c_2(s_1) \partial_{s_2} c_3(s_2) \, dy ds_2 ds_1 dx dt = 0
\]
and hence
\[
\int_0^1 u_1(x, t, y, s_1, s_2) \partial_{s_2} c_3(s_2) \, ds_2 = 0
\]
for all \(c_3 \in C_\infty(0, 1)\), a.e. in \(\Omega_T \times Y \times (0, 1)\). This is the weak form of (130) and means that \(u_1\) is independent of \(s_2\).

Next, we study (126) for the test functions
\[
v(x) = \varepsilon v_1(x) v_2 \left( \frac{x}{\varepsilon} \right), \quad c(t) = c_1(t) c_2 \left( \frac{t}{\varepsilon} \right)
\]
where \(v_1 \in D(\Omega),\ v_2 \in C_\infty(Y) / \mathbb{R},\ c_1 \in D(0, T)\) and \(c_2 \in C_\infty(0, 1)\). We obtain
\[
\int_{\mathcal{Y}_1,2} -u^\varepsilon(x, t) v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) \varepsilon \partial_t c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) + c_1(t) \partial_{s_1} c_2 \left( \frac{t}{\varepsilon_r} \right) + \nabla u^\varepsilon(x, t) \cdot \left( \varepsilon \nabla v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) + v_1(x) \nabla_y v_2 \left( \frac{x}{\varepsilon} \right) \right) c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) \, dx dt = \int_{\mathcal{Y}_1,2} f(x, t, \varepsilon v_1(x) v_2 \left( \frac{x}{\varepsilon} \right) c_1(t) c_2 \left( \frac{t}{\varepsilon} \right) \, dx dt
\]
and when \(\varepsilon\) goes to zero, we get
\[
\int_{\mathcal{Y}_1,2} -u(x, t) v_1(x) v_2(y) c_1(t) \partial_{s_1} c_2(s_1) + a(y, s_1, s_2)(\nabla u(x, t) + \nabla_y u_1(x, t, y, s_1)) \cdot v_1(x) \nabla_y v_2(y) c_1(t) c_2(s_1) \, dy ds_2 ds_1 dx dt = 0.
\]
The periodicity of \( c_2 \) and the variational lemma imply that

\[
\int_Y \left( \int_0^1 a(y, s_1, s_2) \, ds_2 \right) \left( \nabla u(x, t) + \nabla_y u_1(x, t, y, s_1) \right) \cdot \nabla_y v_2(y) \, dy = 0
\]

for all \( v_2 \in H^1_u(Y) / \mathbb{R} \) a.e. in \( \Omega_T \times (0, 1) \). This is the weak form of (131). ■

**Remark 77** Homogenization of linear parabolic equations with oscillations in both space and time was first investigated by means of asymptotic expansions by Bensoussan et al. in [BLP]. This is also studied in [PrTe] by Profeti and Terreni, where a different proof is provided. The homogenization of monotone nonlinear problems, with two scales in space and time, respectively, is proven by means of G-convergence by Svanstedt in [Sva1] and [Sva2]. Similar results can also be found in [Pan]. For corrector results for linear parabolic equations with oscillation in space only, we refer to [BFM] by Brahim-Otsmane et al. Such results for problems with oscillations also in time have been proven by means of two-scale convergence methods by Holmbom in [Hol2], and with H-convergence techniques by Dall’Aglio and Murat in [DaMu]. The corresponding result for nonlinear problems is presented by Svanstedt in [Sva3]. Equations of this kind are also considered by Wellander in [Wel]. In [HSW] Holmbom, Svanstedt and Wellander proved a reiterated homogenization result for linear parabolic operators with two scales of oscillation in space, and different frequencies of oscillation in time, which is the linear correspondence to the result in Section 5.3.2. Some results on nonlinear parabolic problems with one spatial scale and one temporal scale can be found in [NaRa] by Nandakumaran and Rajesh, and in [NgWo1] and [NgWo2] by Nguetseng and Woukeng concerning such problems without periodicity assumptions.
6 Characterization of the $G$-limit for some non-periodic linear operators

So far, we have seen a few different examples where it has turned out to be possible to determine the $G$-limit. In Chapter 3, we came across some cases where the convergence of the sequence $\{a^h\}$ of coefficients to some limit $a$ was strong enough to ensure that $a$ was also the $G$-limit. Parts of Chapters 4 and 5 concerned some linear periodic homogenization problems where $\{a^h\}$ was in general only weakly convergent in some suitable Lebesgue space. For such problems the determination of the $G$-limit required non-trivial techniques, and it turned out that the $G$-limit was not equal to the weak limit of $\{a^h\}$.

Thus, it might be of interest to see if we can find cases in which it is possible to relax the strength of the convergence of $\{a^h\}$ compared to, for example, the situations in Remark 12 and Proposition 18, and where the $G$-limit is still some conventional kind of limit of $\{a^h\}$. Another interesting area to investigate is non-periodic problems. In this chapter, we will study some linear, not necessarily periodic, problems where the coefficients $a^h$ are created by means of some special integral operators. In Section 6.1, we study elliptic equations with such coefficients and give a criterion for when the $G$-limit is equal to the weak $L^2(\Omega)^{N\times N}$-limit of $\{a^h\}$, and in Section 6.2 we do the corresponding for parabolic problems and discuss some open questions.

6.1 Linear elliptic operators

The interest in this section is to study $G$-convergence of the operators associated with the equations

$$-\nabla \cdot (a^h(x) \nabla u^h(x)) = f(x) \quad \text{in } \Omega,$$

$$u^h(x) = 0 \quad \text{on } \partial \Omega,$$

(135)

where $a^h$ is generated by certain integral operators. We start out with a class of coefficients with entries

$$a^h_{ij}(x) = \int_A w^h_{ij}(y) K_{ij}(x, y) \, dy,$$

where $A$ is an open bounded set in $\mathbb{R}^M$ with smooth boundary, $K$ belongs to $L^2(\Omega \times A)^{N\times N}$ and for some $w$ in $L^2(A)^{N\times N}$

$$w^h(y) \rightharpoonup w(y) \quad \text{in } L^2(A)^{N\times N}.$$
This can be interpreted as a so-called Hilbert-Schmidt integral operator defined by $K$ and acting on $w^h$; see Remark 78. By the compactness of such operators, we have strong convergence of $\{a^h\}$ in $L^2(\Omega)^{N \times N}$ and hence

$$a^h_{ij}(x) \rightarrow a_{ij}(x) = \int_A w_{ij}(y) K_{ij}(x,y) \, dy \quad \text{in } L^2(\Omega).$$

If $\{a^h\}$ belongs to $\mathcal{M}(\alpha, \beta, \Omega)$, Proposition 18 now gives that $a$ is the $G$-limit.

If the convergence of $\{w^h\}$ is disturbed in the sense that we let $w^h$ depend on $x$, the result becomes more difficult to predict. In the next two subsections, we will investigate $G$-convergence for sequences $\{a^h\}$ with the entries

$$a^h_{ij}(x) = \int_A w^h_{ij}(x,y) K_{ij}(x,y) \, dy \quad \text{(136)}$$

for some different choices of $\{w^h\}$.

**Remark 78** A compact operator $T : X \rightarrow Z$ from one Banach space to another has the property that, if $X$ is reflexive, it creates a strongly convergent sequence from a weakly convergent one when acting on it; that is, if

$$w^h \rightharpoonup w \quad \text{in } X,$$

then

$$Tw^h \rightarrow Tw \quad \text{in } Z.$$

Hilbert-Schmidt integral operators provide an example of a class of compact operators (see [Alt] 8.9). An operator $T : L^p(A) \rightarrow L^q(\Omega)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, of this kind is defined by

$$(T w)(x) = \int_A w(y) K(x,y) \, dy$$

where $K \in L^q(\Omega \times A)$. Since $L^p(A)$ is reflexive and $T$ is compact, we have that if

$$w^h(y) \rightharpoonup w(y) \quad \text{in } L^p(A)$$

then

$$(T w^h)(x) \rightarrow (T w)(x) \quad \text{in } L^q(\Omega),$$

that is,

$$\int_A w^h(y) K(x,y) \, dy \rightarrow \int_A w(y) K(x,y) \, dy \quad \text{in } L^q(\Omega). \quad \text{(137)}$$
6.1.1 Some numerical experiments

The questions concerning ways in which we can relax the strength of the convergence of \( \{a^h\} \) until there appear substantial difficulties in deciding the \( G \)-limit can be illuminated by suitable choices of \( \{w^h\} \) and \( K \) in (136). We study some two-dimensional examples, where we choose \( \Omega = (1,3)^2 \), \( A = Y = (0,1)^2 \) and \( f(x) = 100x_1 \).

Example 1

First, we consider coefficients of the form

\[
a^h_{ij}(x) = \int_Y w^h_{ij}(y) K_{ij}(x,y) \, dy,
\]

which can be associated with Hilbert-Schmidt integral operators. We let

\[
w^h(y) = \begin{pmatrix}
2 + \sin(2\pi h(y_1 + y_2)) & \frac{3}{2} + \sin(2\pi h(y_1 + y_2)) \\
\frac{3}{2} + \sin(2\pi h(y_1 + y_2)) & 2 + \sin(2\pi h(y_1 + y_2))
\end{pmatrix}
\]

and

\[
K_{ij}(x,y) = 2 + \sin(x_1 + x_2 + y_1 + y_2).
\]

Obviously,

\[
w^h(y) \to w = \begin{pmatrix}
2 & \frac{3}{2} \\
\frac{3}{2} & 2
\end{pmatrix}
\]

in \( L^2(Y)^{2 \times 2} \),

and hence

\[
a^h_{ij}(x) \to a_{ij}(x) = \int_Y w_{ij}(y) K_{ij}(x,y) \, dy \quad \text{in } L^2(\Omega);
\]

see (137). By Proposition 18, this means that we have found the \( G \)-limit. In Figure 27 below, we see \( a_{12} \) and the solution \( u \) obtained for the limit coefficient \( a \).

![Figure 27. \( a_{12} \) and the solution \( u \).](image_url)
Example 2

Next, we study an example where we allow $w^h$ to depend on $x$ as well. For

$$a^h_{ij}(x) = \int_Y w^h_{ij}(x, y) K_{ij}(x, y) \, dy,$$  \hspace{1cm} (141)

with

$$w^h(x, y) = \begin{pmatrix} 2 + \sin(2\pi h(x_1 + x_2) + y_1 + y_2) \\ \frac{3}{2} + \sin(2\pi h(x_1 + x_2) + y_1 + y_2) \end{pmatrix},$$ \hspace{1cm} (142)

we obtain

$$w^h(x, y) \rightharpoonup w = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} \quad \text{in } L^2(\Omega \times Y)^{2 \times 2}. \hspace{1cm} (143)$$

This means that we still have weak convergence of $\{w^h\}$ to the same matrix as in (139), now in the space $L^2(\Omega \times Y)^{2 \times 2}$, but since we can no longer interpret the coefficient as the result of the action of a Hilbert-Schmidt integral operator, we are not guaranteed strong convergence of $\{a^h\}$. However, $\{a^h\}$ still converges weakly, that is, with $K$ defined by (138), we obtain

$$a^h_{ij}(x) \rightharpoonup \int_Y w_{ij}(x, y) K_{ij}(x, y) \, dy \quad \text{in } L^2(\Omega),$$ \hspace{1cm} (144)

where the limit coincides with $a_{ij}$ in (140) for our choice of $w^h$ in (142). Now the question is: does this mean that we obtain the same $G$-limit, that is, is the $G$-limit in this case the weak $L^2(\Omega)^{N \times N}$-limit of $\{a^h\}$ obtained in (144)?

In Figure 28, to the left, we have the entry $a^h_{12}$ of $a^h$ where we can see the oscillations in the $x_1$ and $x_2$ variables.

![Figure 28. $a^h_{12}$ and the solution $u^h$ for $h = 10$.](image)
The other entries have essentially the same pattern of oscillations. To the right we see the solution $u^h$ to (135) for $a^h$ given by (141). It apparently does not coincide with $u$ in Figure 27, the solution obtained for the coefficient $a$, and the same phenomenon will remain for larger $h$. Thus, it seems that the $G$-limit differs from $a$ in this case, and hence from the weak limit. Obviously, the oscillations in $x_1$ and $x_2$ have affected the $G$-limit.

**Example 3**

Finally, we consider a second example where $w^h$ is dependent on $x$, namely

$$w^h(x, y) = \left( \frac{2 + \sin(2\pi h(x_1 + x_2) + y_1 + y_2)}{3} - \sin(2\pi h(x_1 + x_2) + y_1 + y_2) \right) \left( \frac{3}{2} - \sin(2\pi h(x_1 + x_2) + y_1 + y_2) \right) 2 + \sin(2\pi h(x_1 + x_2) + y_1 + y_2) \right)$$  \hspace{1cm} (145)

and with the same $K$ as before. We obtain the same weak limit of $\{w^h\}$ given in (143), and hence the same weak limit of $\{a^h\}$, which is identical to the limits in (140) and (144). In Figure 29, we see $a^h_{12}$ together with the solution $u^h$.

![Figure 29](image)

Figure 29. $a^h_{12}$ and the solution $u^h$ for $h = 10$. The oscillations in the coefficient have the same magnitude as in the last example. However, in this case they do not appear to have any effect on the solution. We get a solution very much like $u$, which indicates that the $G$-limit is again the function $a$ obtained already in (140); that is, for this example the $G$-limit appears to be the weak $L^2(\Omega)^{N \times N}$-limit of $\{a^h\}$. Thus, the two problems where we have dependence of $x$ in $w^h$ do not seem to lead to the same $G$-limit, even though the weak limit for the respective sequences of coefficients coincides. In Example 3, the $G$-limit appears to be the function $a$ in (140) whereas in Example 2 we have a deviation from $a$. In the next section, we will explain this phenomenon in theoretical terms.
6.1.2 A theoretical result

Through a couple of examples, we have shown that the $G$-limit might be affected by introducing the dependence of $x$ in $w^h$ even when the weak limit of $\{w^h\}$ remains the same. Below, we provide a result that distinguishes some cases where the determination of the $G$-limit is straightforward.

**Theorem 79** Let $\{a^h\}$ be a sequence in $\mathcal{M}(\alpha, \beta, \Omega)$ such that

$$a^h_{ij}(x) = \int_A w^h_{ij}(x,y)K_{ij}(x,y) \, dy,$$

where $w^h, K \in C^1(\bar{\Omega} \times \bar{A})^{N \times N}$ and

$$w^h(x,y) \rightarrow w(x,y) \quad \text{in} \quad L^2(\Omega \times A)^{N \times N}$$

for some $w \in H^1(\Omega \times A)^{N \times N}$. Assume also that for $i = 1, 2, \ldots, N$

$$\sum_{j=1}^N \int_A \partial_{x_j} w^h_{ij}(x,y)K_{ij}(x,y) \, dy \to 0 \quad \text{in} \quad L^2(\Omega)$$

and that $b \in \mathcal{M}(\alpha, \frac{\beta^2}{\alpha}, \Omega)$ has the entries

$$b_{ij}(x) = \int_A w_{ij}(x,y)K_{ij}(x,y) \, dy.$$

Then $\{a^h\}$ $G$-converges to $b$.

**Proof.** Since $\{a^h\}$ belongs to $\mathcal{M}(\alpha, \beta, \Omega)$, we know that there exists at least a subsequence that $G$-converges. To identify the $G$-limit, we want to study the limit process for the problem (135), i.e.

$$\sum_{i,j=1}^N \int_\Omega a^h_{ij}(x)\partial_{x_j} u^h(x)\partial_{x_i} v(x) \, dx = \int_\Omega f(x)v(x) \, dx$$

for all $v \in D(\Omega)$, for such a sequence. Integrating by parts, the left-hand side reads

$$-\sum_{i,j=1}^N \int_\Omega u^h(x)\partial_{x_j} a^h_{ij}(x)\partial_{x_i} v(x) + u^h(x)a^h_{ij}(x)\partial^2_{x_i x_j} v(x) \, dx.$$ 

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Inserting (146), we get
\[
- \sum_{i,j=1}^{N} \int_{\Omega} \int_{A} u^h(x) \left( \partial_{x_j} w_{ij}^h(x, y) K_{ij}(x, y) + w_{ij}^h(x, y) \partial_{x_j} K_{ij}(x, y) \right) \partial_{x_i} v(x) + \\
\quad u^h(x) w_{ij}^h(x, y) K_{ij}(x, y) \partial_{x_i}^2 v(x) \, dy \, dx,
\]
where we use (148) and the convergence properties of \{w^h\} and \{u^h\} to obtain, after letting \(h \to \infty\),
\[
- \sum_{i,j=1}^{N} \int_{\Omega} \int_{A} u(x) w_{ij}(x, y) \partial_{x_j} K_{ij}(x, y) \partial_{x_i} v(x) + \\
\quad u(x) w_{ij}(x, y) K_{ij}(x, y) \partial_{x_i}^2 v(x) \, dy \, dx.
\]
Integrating by parts in the second term, we have
\[
\sum_{i,j=1}^{N} \int_{\Omega} \int_{A} -u(x) w_{ij}(x, y) \partial_{x_j} K_{ij}(x, y) \partial_{x_i} v(x) + \left( \partial_{x_j} u(x) w_{ij}(x, y) K_{ij}(x, y) + \\
\quad u(x) \partial_{x_j} w_{ij}(x, y) K_{ij}(x, y) + u(x) w_{ij}(x, y) \partial_{x_j} K_{ij}(x, y) \right) \partial_{x_i} v(x) \, dy \, dx,
\]
where the first and the last term cancel out and we are left with
\[
\sum_{i,j=1}^{N} \int_{\Omega} \int_{A} (w_{ij}(x, y) K_{ij}(x, y) \partial_{x_j} u(x) + u(x) \partial_{x_j} w_{ij}(x, y) K_{ij}(x, y)) \partial_{x_i} v(x) \, dy \, dx.
\]
By rearrangement, this can be written
\[
\sum_{i,j=1}^{N} \left( \int_{\Omega} b_{ij}(x) \partial_{x_j} u(x) \partial_{x_i} v(x) \, dx + \int_{\Omega} \int_{A} u(x) \partial_{x_j} w_{ij}(x, y) K_{ij}(x, y) \partial_{x_i} v(x) \, dy \, dx \right).
\]
This means that it remains to be proven that
\[
\sum_{i,j=1}^{N} \int_{\Omega} \int_{A} u(x) \partial_{x_j} w_{ij}(x, y) K_{ij}(x, y) \partial_{x_i} v(x) \, dy \, dx = 0. \quad (149)
\]
By assumption, we have
\[
\sum_{j=1}^{N} \int_{A} \partial_{x_j} w_{ij}^h(x, y) K_{ij}(x, y) \, dy \to 0 \quad \text{in } L^2(\Omega)
\]
and hence
\[ \sum_{i,j=1}^{N} \int_{\Omega} \int_{A} \partial_{x_j} w_{ij}^h(x,y) K_{ij}(x,y) u(x) \partial_{x_i} v(x) \ dy \ dx \to 0 \]
for all \( v \in D(\Omega) \). Integration by parts gives us
\[ \sum_{i,j=1}^{N} \int_{\Omega} \int_{A} w_{ij}^h(x,y) \partial_{x_j} K_{ij}(x,y) u(x) \partial_{x_i} v(x) + w_{ij}^h(x,y) v(x) \partial_{x_j}^2 \partial_{x_i} \partial_{x_j} u(x) \ dy \ dx \to 0. \]

Using (147) in the left-hand side, we have
\[ \sum_{i,j=1}^{N} \int_{\Omega} \int_{A} w_{ij}(x,y) \partial_{x_j} K_{ij}(x,y) u(x) \partial_{x_i} v(x) + w_{ij}(x,y) v(x) \partial_{x_j}^2 \partial_{x_i} \partial_{x_j} u(x) \ dy \ dx = 0. \]

A final integration by parts results in
\[ \sum_{i,j=1}^{N} \int_{\Omega} \int_{A} \partial_{x_j} w_{ij}(x,y) K_{ij}(x,y) u(x) \partial_{x_i} v(x) \ dy \ dx = 0 \]
and (149) is proven. Hence \( b \) is the \( G \)-limit for the chosen subsequence, and since the same result will follow for any convergent subsequence the entire sequence \( G \)-converges to \( b \). ■

It is now clear how we could get results that indicated different \( G \)-limits in the two examples with \( u^h \) dependent on \( x \) in the previous section. In Example 2 the criterion (148) is not fulfilled, whereas in Example 3 it is obviously satisfied since the derivatives \( \partial_{x_j} w_{ij}^h \) cancel out when summing over \( j \) for the choice of \( u^h \) given in (145). Thus, the \( G \)-limit is truly equal to the weak limit of \( \{ a^h \} \) in the Example 3.
6.2 Linear parabolic operators

In this section we will investigate sequences of equations of the form

\[
\partial_t u_h(x, t) - \nabla \cdot (a^h(x, t) \nabla u_h(x, t)) = f(x, t) \quad \text{in } \Omega_T,
\]

\[
u^h(x, 0) = u^0(x) \quad \text{in } \Omega,
\]

\[
u^h(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T),
\]

with respect to $G$-convergence, where the coefficients are created in a way similar to that in the preceding section. A slight modification of the operators studied there, to fit the evolution setting, yields

\[
a_{ij}^h(x, t) = \int_A w_{ij}^h(x, t, y) K_{ij}(x, t, y) \, dy.
\]

If $\{w^h\}$ and $K$ are chosen such that $\{a^h\}$ belongs to $\mathcal{M}(\alpha, \beta, \Omega_T)$, the existence of a $G$-convergent subsequence is evident by Theorem 25. Moreover, if $w^h$ is independent of $x$ and $t$ and

\[
w_{ij}^h(y) \rightharpoonup w_{ij}(y) \quad \text{in } L^2(A),
\]

then, for any $K \in L^2(\Omega \times A)$,

\[
a_{ij}^h(x, t) \rightharpoonup \int_A w_{ij}(y) K_{ij}(x, t, y) \, dy \quad \text{in } L^2(\Omega_T)
\]

and hence the entire sequence $G$-converges to the unique limit

\[
b(x, t) = \int_A w_{ij}(y) K_{ij}(x, t, y) \, dy.
\]

In the following sections, we will investigate the limit behavior of equations of the type in (150) when $\{w^h\}$ may depend also on $x$ and $t$.

6.2.1 A theoretical result

Below, we prove a mathematical criterion for the parabolic case, similar to the one given in Section 6.1.2, for when the $G$-limit is trivial to determine, namely for when it coincides with the weak $L^2(\Omega_T)^{N \times N}$-limit of $\{a^h\}$. We note that the comparison theorem given in Section 4.3 provides a shortcut, if we make a certain continuity assumption.
Theorem 80 Assume that \( \{a^h\} \) belongs to \( \mathcal{M}(\alpha, \beta, \Omega) \) for every fixed \( t \in (0, T) \) and that
\[
\left| \left( a^h(x, t) - a^h(x, t') \right) \xi \right| \leq B(t - t')(1 + |\xi|) \tag{152}
\]
for all \( \xi \in \mathbb{R}^N \) and all \( t', t \) such that \( 0 < t' < t < T \), where \( B : \mathbb{R}_+ \to \mathbb{R}_+ \) is an increasing continuous function such that \( B(t) \to 0 \) as \( t \to 0_+ \). Furthermore, let \( \{a^h\} \) be a sequence such that
\[
a^h_{ij}(x, t) = \int_A w^h_{ij}(x, t, y)K_{ij}(x, t, y) \, dy,
\]
where \( w^h \), \( K \in C^1(\bar{\Omega}_T \times \bar{A})^{N \times N} \) and for some \( w(t) \in H^1(\Omega \times A)^{N \times N} \)
\[
w^h(x, t, y) \rightharpoonup w(x, t, y) \text{ in } L^2(\Omega \times A)^{N \times N} \tag{153}
\]
for every fixed \( t \in (0, T) \). Assume also that
\[
\sum_{j=1}^{N} \int_A \partial_{x_j}w^h_{ij}(x, t, y)K_{ij}(x, t, y) \, dy \to 0 \text{ in } L^2(\Omega) \tag{154}
\]
for all \( t \in (0, T) \). Then \( \{a^h\} \) \( G \)-converges in the sense of parabolic operators to
\[
b_{ij}(x, t) = \int_A w_{ij}(x, t, y)K_{ij}(x, t, y) \, dy. \tag{155}
\]

Proof. From the assumption that \( \{a^h\} \) is in \( \mathcal{M}(\alpha, \beta, \Omega) \) for every fixed \( t \in (0, T) \), it follows that it belongs to \( \mathcal{M}(\alpha, \beta, \Omega_T) \). This means that, up to a subsequence, we have \( G \)-convergence in the parabolic sense to some \( b \) according to Theorem 25. By assumption, \( \{a^h\} \) fulfills (152) and hence we know from Theorem 29 that it \( G \)-converges in the elliptic sense, up to a subsequence, for every fixed \( t \) to some limit \( b'(t) \). Thus, by Theorem 28, \( b \) coincides with \( b' \). Moreover, since \( \{a^h\} \) agrees with (152), (153) and (154) it follows, in view of Theorem 79, that \( b \) is given by (155). \( \blacksquare \)

This result covers possible heavy oscillations in \( x \) and parameter dependence of \( t \). Thus, we can handle a large class of parabolic problems as well. However, oscillations in \( t \) require some more investigations. This will be explored in the next section.
6.2.2 Some further results and open questions

Let us now introduce temporal oscillations in $w^h$ from Example 2 in Section 6.1.1. We choose $w^h$ as the matrix given in (142) with the second term in each component multiplied by $\sin(2\pi ht)$, i.e. we have for instance

$$w^h_{11}(x, t, y) = 2 + \sin(2\pi h(x_1 + x_2) + y_1 + y_2)\sin(2\pi ht).$$  \hspace{1cm} (156)

We still let $\Omega = (1, 3)^2$, $A = Y = (0, 1)^2$ and $f(x, t) = 100x_1$. If we choose $u^0(x) = 0$ and solve problem (150) for this choice of $a^h$, we obtain the solution in Figure 30 below for $t = 1$. In contrast to the corresponding elliptic case, the $G$-limit appears to be close or even identical to the weak limit $a$ obtained in (144).

![Figure 30. The solution $u^h$ for $h = 20$.](image)

For this example, the criterion (154) is not fulfilled for any fixed $t$ such that $2t \not\in \mathbb{Z}$. Actually, since we have introduced temporal oscillations, condition (152) is not satisfied and thus Proposition 80 is no longer applicable.

The temporal oscillations obviously have an effect in this example, and it seems that they, so to speak, have neutralized or at least significantly reduced the effect of the oscillations in space. Thus, it might be of interest to investigate whether we could find some alternative to the criterion (154). We try multiplying the sum in (154) with test functions $v$, depending also on $t$ and integrating over $\Omega_T$. For different choices of $v$ in $C(\overline{\Omega_T})$, we obtain

$$\sum_{j=1}^{\infty} \int_{\Omega_T} \int_Y \partial_x w^h_{ij}(x, t, y)K_{ij}(x, t, y) \, dy \, v(x, t) \, dx \, dt \to 0.$$

Thus, it seems that we might get something new out of integrating also over the time interval.
A result that is in line with this is given below. It shows that there are criteria for time-dependent problems, corresponding to (147) and (148), but where the convergences take place in $L^2(\Omega_T \times A)^{N \times N}$ and $L^2(\Omega_T)$, respectively. This result also means that we can omit the condition (152).

**Theorem 81** Let $\{a^h\}$ be a sequence in $\mathcal{M}(\alpha, \beta, \Omega_T)$ such that

$$a^h_{ij}(x, t) = \int_A w^h_{ij}(x, t, y) K_{ij}(x, t, y) \, dy,$$

where $w^h, K \in C^1(\Omega_T \times \tilde{A})^{N \times N}$ and

$$w^h(x, t, y) \rightarrow w(x, t, y) \quad \text{in } L^2(\Omega_T \times A)^{N \times N}$$

for some $w \in L^2(0, T; H^1(\Omega \times A))^{N \times N}$. Assume also that

$$\sum_{j=1}^N \int_A \partial_{x_j} w^h_{ij}(x, t, y) K_{ij}(x, t, y) \, dy \rightarrow 0 \quad \text{in } L^2(\Omega_T) \quad (157)$$

and that $b \in \mathcal{M}(\alpha, \beta^2/\alpha, \Omega_T)$ has the entries

$$b_{ij}(x, t) = \int_A w^h_{ij}(x, t, y) K_{ij}(x, t, y) \, dy. \quad (158)$$

Then $\{a^h\}$ $G$-converges to $b$.

**Proof.** The proof can be carried out by combining the methods used in [FHOSi1] and [Sil2], which treat the case when $w = 1$, and the technique of the proof for the elliptic case in Proposition 79. □

Actually, for our example, we do not obtain the weak convergence in $L^2(\Omega_T)$ given in (157). This is realized if in place of the test function, we try for example with the sequence $\{v^h\}$ of functions

$$v^h(x, t) = x_1^2 + \frac{1}{h} \sin(2\pi h(x_1 + x_2 + t)), \quad (159)$$

which is strongly convergent in $L^2(\Omega_T)$ towards $x_1^2$. Indeed, we have

$$\lim_{h \to \infty} \sum_{j=1}^N \int_{\Omega_T} \int_{\tilde{A}} \partial_{x_j} w^h_{ij}(x, t, y) K_{ij}(x, t, y) \, dy \, v^h(x, t) \, dx \, dt \approx 0.655,$$

and evidently (157) is not fulfilled.

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So what can we say about this particular example? By Proposition 3.2 in [AlBr], for this choice of \( w \) and \( K \) we obtain that

\[
\left\| \sum_{j=1}^{N} \int_{A} \partial_{x_j} w_{ij}^{h}(x, t, y) K_{ij}(x, t, y) \, dy \right\|_{L^2(0, T; H^{-1}(\Omega))} \leq C,
\]

and hence we have at least weak* convergence in \( L^2(0, T; H^{-1}(\Omega)) \), up to a subsequence. Different choices of test functions \( v \in L^2(0, T; H^1_{0}(\Omega)) \) give

\[
\sum_{j=1}^{N} \int_{\Omega_T} \int_{A} \partial_{x_j} w_{ij}^{h}(x, t, y) K_{ij}(x, t, y) \, dy \, v(x, t) \, dxdt \to 0, \quad (160)
\]

which indicates that the limit is 0, that is

\[
\sum_{j=1}^{N} \int_{A} \partial_{x_j} w_{ij}^{h}(x, t, y) K_{ij}(x, t, y) \, dy \xrightarrow{\ast} 0 \quad \text{in} \quad L^2(0, T; H^{-1}(\Omega)). \quad (161)
\]

This should mean that for any fixed test function in \( L^2(0, T; H^1_{0}(\Omega)) \) we obtain (160), but if we multiply by a sequence that is weakly convergent in \( L^2(0, T; H^1_{0}(\Omega)) \) we have a product of two only weakly convergent sequences and thus there is no obvious way to predict the limit.

If we perform the proof of Proposition 81, we see that what we really need is that

\[
\sum_{j=1}^{N} \int_{\Omega_T} \int_{A} \partial_{x_j} w_{ij}^{h}(x, t, y) K_{ij}(x, t, y) \, dy \, \partial_{x_j} v(x) c(t) u^{h}(x, t) \, dxdt \to 0, \quad (162)
\]

and thus the convergence (161) is generally not enough since the other sequence in (162) is weakly convergent in \( L^2(0, T; H^1_{0}(\Omega)) \). On the other hand it is now clear that if we had strong convergence in \( L^2(0, T; H^{-1}(\Omega)) \) everything would work—that is, if

\[
\sum_{j=1}^{N} \int_{A} \partial_{x_j} w_{ij}^{h}(x, t, y) K_{ij}(x, t, y) \, dy \to 0 \quad \text{in} \quad L^2(0, T; H^{-1}(\Omega)). \quad (163)
\]

Thus, we have an alternative to the key criterion (157). This is not fulfilled in our example, however. To see this, we can employ some sequence of test
functions $v^h$, which is weakly convergent in $L^2(0,T;H^1_0(\Omega))$. It remains an open question whether conditions (157) and (163) could be replaced by some weaker alternative. One key to this may be contained in the properties of the specific sequence of solutions $u^h$ and the relation between $w^h$ and $u^h$, which follows from the fact that $u^h$ solves a PDE where $w^h$ generates the oscillations in the coefficient matrix.

**Remark 82** The examples we have studied in this section can also be regarded as quasiperiodic homogenization problems. If we replace $ht$ in (156) with $hr$, $r > 2$, it is possible to prove that $b$ is actually given by (158).

**Remark 83** The numerical experiments have been carried out on powerful computers. Still, due to the strong oscillations the results are not completely stable and hence should be regarded as a source of inspiration for an experiment of thought rather than really reliable computations. This also emphasizes the importance of further theoretical investigations.

**Remark 84** The considerations in this chapter open up new approaches in the study of convergence for operators without periodicity assumptions, which could also be regarded as non-periodic homogenization. What would happen if the criterion (148) is not fulfilled and we for instance have

$$\sum_{j=1}^{N} \int_A \partial_{x_j} w^h_{ij}(x,y)K_{ij}(x,y) \, dy \rightarrow g_i(x) \quad \text{in} \ L^2(\Omega)?$$

Could the corresponding limit term then be integrated in the coefficient in an elliptic equation and hence be a part of the $G$-limit? Or might it be possible to obtain a different limit equation with a so-called strange term? In that case, the limit problem would not give the $G$-limit, but if it was well-posed we could still obtain a solution that approximates $u^h$ for $h$ large. There are clearly many challenges ahead in continuation of research along these lines, the roots of which can be found in [Sil1], [Sil2] and [HOS].
References


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Appendix

Below is a list of some fundamental definitions and theorems that are used in the thesis. If nothing else is stated, $\Omega$ is open and bounded with at least Lipschitz boundary.

**Definition 85 (Strong convergence)** A sequence $\{u^h\}$ in a normed space $X$ is said to converge strongly, or in norm, to $u$ if

$$\|u^h - u\|_X \to 0.$$  

This is denoted by

$$u^h \to u \quad \text{in } X.$$  

**Definition 86 (Weak convergence)** A sequence $\{u^h\}$ in a normed space $X$ is said to converge weakly to $u$ if

$$F(u^h) \to F(u) \quad \text{for all } F \in X'.$$

We denote this by

$$u^h \rightharpoonup u \quad \text{in } X.$$  

**Definition 87 (Weak* convergence)** A sequence $\{F^h\}$ in the dual $X'$ of a normed space $X$ is said to converge weakly* to $F$ if

$$F^h(u) \to F(u) \quad \text{for all } u \in X.$$  

We use the notation

$$F^h \rightharpoonup^* F \quad \text{in } X'.$$

**Theorem 88** Let $X$ be a reflexive Banach space. Then every bounded sequence $\{u^h\}$ in $X$ has a weakly convergent subsequence.

**Proof.** See [Alt], 5.7. Satz. ■

**Theorem 89** Let $X$ be a separable normed space. Then every bounded sequence $\{F^h\}$ in $X'$ has a weakly* convergent subsequence.

**Proof.** See [ZeiIIA], Theorem 21.E. ■
Theorem 90 \(L^p(\Omega)\) is a Banach space for \(1 \leq p \leq \infty\). Furthermore, it is reflexive for \(1 < p < \infty\) and separable for \(1 \leq p < \infty\).

Proof. See Theorems 2.7.1, 2.8.1 and 2.10.1 in [Kuf]. ■

Theorem 91 For \(1 \leq p \leq \infty\), \(W^{1,p}(\Omega)\) is a Banach space. It is reflexive for \(1 < p < \infty\) and separable for \(1 \leq p < \infty\). Furthermore, the space \(W_0^{1,p}(\Omega)\) is reflexive for \(1 < p < \infty\), and for \(1 \leq p \leq \infty\) it is separable.

Proof. See Theorems 5.4.2, 5.2.2, 5.2.4 and 5.4.4 in [Kuf]. ■

Theorem 92 For \(1 \leq p < \infty\), the dual of \(L^p(\Omega)\) can be identified with \(L^q(\Omega)\), where \(\frac{1}{p} + \frac{1}{q} = 1\). Moreover, \((L^1(\Omega))^\prime\) can be identified with \(L^\infty(\Omega)\).

Proof. See Theorems 2.9.1 and 2.11.8 in [Kuf]. ■

Theorem 93 (Rellich embedding theorem) Let \(1 \leq p < \infty\). If \(u^h \rightarrow u\) in \(W^{1,p}(\Omega)\), then \(u^h \rightarrow u\) in \(L^p(\Omega)\).

Proof. See [Alt], A 5.4. ■

Theorem 94 (Cauchy-Schwarz inequality) For any \(u, v\) in a Hilbert space \(H\) it holds that
\[
|(u,v)_H| \leq \|u\|_H \|v\|_H.
\]

Proof. See Lemma 3.2-1 in [Kre]. ■

Theorem 95 (Hölder’s inequality) Let \(u \in L^p(\Omega)\) and \(v \in L^q(\Omega)\), where \(1 < p < \infty\), \(\frac{1}{p} + \frac{1}{q} = 1\) and \(\Omega\) is a non-empty, measurable set in \(\mathbb{R}^N\). Then
\[
\left| \int_\Omega u(x)v(x) \, dx \right| \leq \int_\Omega |u(x)v(x)| \, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.
\] (164)

Moreover, if \(u \in L^1(\Omega)\) and \(v \in L^\infty(\Omega)\) \((164)\) holds true for \(p = 1\) and \(q = \infty\).

Proof. See [ZeiIIA], Proposition 18.13. ■
Theorem 96 (Poincaré’s inequality) For any $u \in H^1_0(\Omega)$, where $\Omega$ is an open bounded set in $\mathbb{R}^N$, there is a positive constant $C$ such that

$$\int_{\Omega} u(x)^2 \, dx \leq C \int_{\Omega} |\nabla u(x)|^2 \, dx$$

where $C$ depends only on $\Omega$.

**Proof.** See 4.10 in [Alt].

Theorem 97 (Poincaré-Wirtinger inequality) Let $1 \leq p < \infty$. Then there exists a positive constant $C$ such that

$$\|u - M_\Omega(u)\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)^N}$$

for every $u \in W^{1,p}(\Omega)$, where $M_\Omega(u)$ denotes the integral mean value of $u$ over $\Omega$.

**Proof.** See p. 194 in [McO].

Theorem 98 (Riesz representation theorem) Let $F$ be a bounded linear functional on the Hilbert space $H$ (i.e. let $F \in H'$). Then there is a unique element $u \in H$ such that

$$F(v) = (u, v)_H \quad \text{for every } v \in H$$

with

$$\|F\|_{H'} = \|u\|_H.$$

**Proof.** See [Mad], Theorem 13.

Theorem 99 (Lax-Milgram theorem) Let $H$ be a Hilbert space and let $a$ be a continuous bilinear form on $H \times H$ satisfying

$$|a(u, v)| \leq \beta \|u\|_H \|v\|_H$$

and

$$a(u, u) \geq \alpha \|u\|^2_H$$

for all $u, v \in H$ and some $\alpha > 0$, and let $F \in H'$. Then there is a unique $u \in H$ such that

$$a(u, v) = \langle F, v \rangle_{H', H}$$

for all $v \in H$.

**Proof.** See [CiDo], Theorem 4.6.
Theorem 100 (Variational lemma) Let $\Omega$ be a non-empty open set in $\mathbb{R}^N$, let $u \in L^2(\Omega)$ and assume that

$$\int_{\Omega} u(x)v(x) \, dx = 0 \quad \text{for every } v \in C_0^\infty(\Omega).$$

Then

$$u(x) = 0 \quad \text{for almost every } x \in \Omega.$$

Proof. See [ZeiIIA], Proposition 18.2. 

Theorem 101 (Lebesgue’s generalized majorized convergence theorem)
Let $\Omega \subset \mathbb{R}^N$ be a measurable set and $f^h : \Omega \to \mathbb{R}$ be measurable for all $h$ and assume that $\{f^h\}$ converges to $f$ a.e. in $\Omega$. Assume also that there are integrable functions $g^h : \Omega \to \mathbb{R}$ such that

$$|f^h(x)| \leq g^h(x) \quad \text{for a.e. } x \in \Omega,$$

$\{g^h\}$ converges to $g$ a.e. in $\Omega$ and

$$\int_{\Omega} g^h(x) \, dx \to \int_{\Omega} g(x) \, dx.$$

Then

$$\lim_{h \to \infty} \int_{\Omega} f^h(x) \, dx = \int_{\Omega} \lim_{h \to \infty} f^h(x) \, dx.$$

Proof. See [ZeiIIA], (19a) in the appendix. 

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